Single Exponential FPT Algorithm for Interval Vertex Deletion and Interval Completion Problem

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Abstract

Let G be an input graph with n vertices and m edges and let k be a fixed parameter. We provide a single exponential FPT algorithm with running time $O(c^k n(n+m))$, $c = min\{18, k\}$ that turns graph G into an interval graph by deleting at most k vertices from G. This solves an open problem posed by Marx [19].

We also provide a single exponential FPT algorithm with running time $O(c^k n(n+m))$, $c = min\{17, k\}$ that turns G into an interval graph by adding at most k edges. The first FPT algorithm with run time $O(k^{2k} n^3 m)$ appeared in STOC 2007 [24]. Our algorithm is the the first single exponential FPT algorithm that improves the running time of the previous algorithm.

The algorithms are based on a structural decomposition of G into smaller subgraphs when G is free from small interval graph obstructions. The decomposition allows us to manage the search tree more efficiently.

1 Introduction

An interval graph is a graph G which admits an interval representation, i.e., a family of intervals I_v , $v \in V(G)$, such that $uv \in E(G)$ if and only if I_u and I_v intersect. Interval graphs have been characterized in many different ways [8, 9, 12, 18]. The following theorem is the best known characterization.

Theorem 1.1 *[18] G is an interval graph if and only if it contains no asteroidal triple and no induced cycle C_k , $k > 3$.*

An asteroidal triple, AT, is an induced subgraph S of G with three none adjacent vertices a, b, c such that for every permutation x, y, z of a, b, c there is a path between x, y outside the neighborhood of z. A graph G is chordal if it does not contain an induced cycle C_k, $k \geq 4$. Cycle C_k in G is induced if it does not have any chord; an edge in G joining two non-adjacent vertices of the cycle.

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The k-interval deletion problem is as follows: Given a graph G and integer k, one asks whether there is a way of deleting at most k vertices from G such that the resulting graph is interval.

The k-interval completion (minimum interval completion) problem is as follows: Given a graph G and integer k, one asks whether there is a way of adding at most k edges to G such that the resulting graph is interval.

Both k-interval deletion problem and k-interval completion problem are known to be NP-hard [10, 16]. The k-chordal completion and k-proper interval completion problem are defined respectively. These problems arise in area such as sparse matrix computations [11], database management [1, 23], computer vision [3], and physical mapping of DNA [11, 13]. Due to their practical applications they have been extensively studied.

A parameterized problem with parameter k and input size x that can be solved by an algorithm with runtime $f(k) \cdot x^{O(1)}$ is called a fixed parameter tractable (FPT) where $f(k)$ is a computable function of k (see [6] for an introduction to fixed parameter tractability and bounded search tree algorithms). An early result related to k-interval completion problem is due to Kaplan, Shamir and Tarjan [15]. They gave an FPT algorithm for k-chordal completion, k-strongly chordal completion, and k-proper interval completion problem. The first FPT algorithm with runtime $O(k^{2k}n^3m)$ for the k-interval completion problem was developed by Villanger, Heggernes, Paul and Telle [24].

The k-interval deletion problem was posed by D.Marx [19]. He considered the k-chordal deleting problem as follows. Given an input graph G and a parameter k, one asks whether there is a way of deleting at most k vertices from G such that the resulting graph becomes chordal. Marx deployed a heavy machinery to obtain an FPT algorithm for k-chordal deletion problem.

In the approximation world, there is no constant approximation algorithm for minimum interval completion problem. The first $O(log^2 n)$-approximation algorithm for minimum interval completion was obtained by Ravi, Agrawal and Klein [21] and then it was improved to an $O(logn \ \ loglogn)$- approximation by Even, Naor, Rao and Schieber [7] and finally to an $O(logn)$-approximation algorithm by Rao and Richa [20]. There are polynomial time algorithms for minimum interval completion on special classes of graphs. The minimum interval completion is polynomial time solvable on trees. Kuo and Wang [17] gave an $O(n^{1.77})$ algorithm minimum interval completion on trees and then it was improved to $O(n)$ algorithm by Diaz, Gibbons, Paterson and Torn [4].

We use deep structural graphs theory analysis to obtain a single exponential FPT algorithms to solve the k-interval deletion problem and k-interval completion problem. We do not use many non-standard terminology and definitions and we refer to a standard text book in graph theory such as [5].

The broad overview of the algorithms :

Suppose G contains only small AT's (See Figure 1) or induced cycles of length at most 8. Then for each small forbidden subgraph Z of G we consider all the possible ways of deleting (adding) one vertex (a few edges when Z is an induced cycle) from (to) Z and hence we can

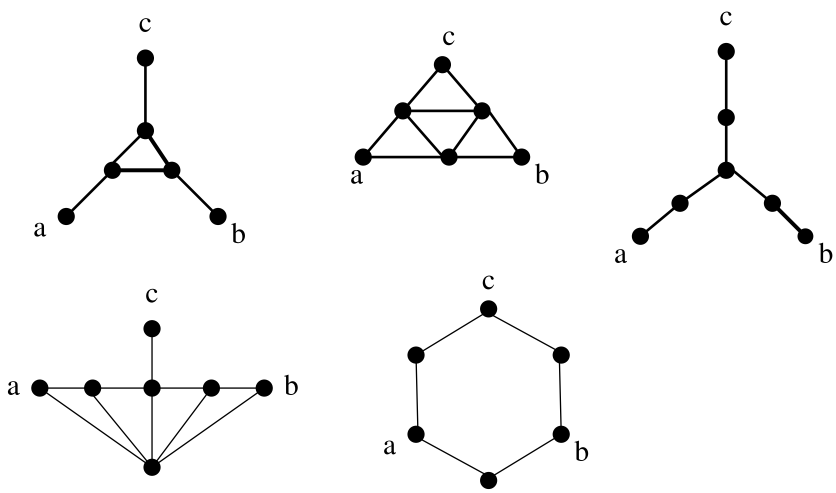

Figure 1: Small ATs

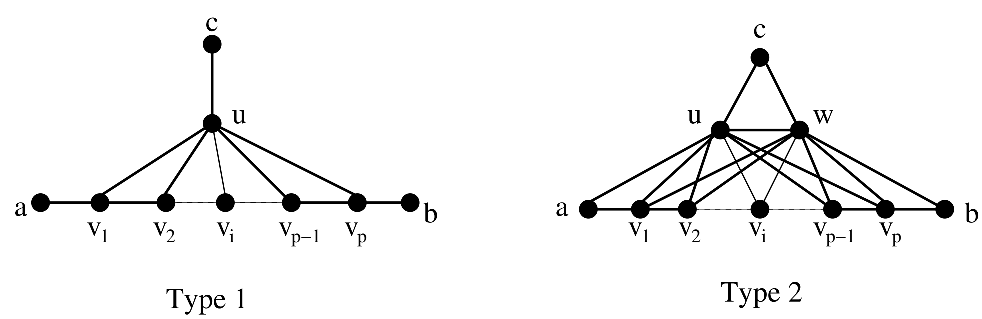

Figure 2: Big ATs

follow a search three with at most 8 branches. This is standard technique in developing FPT algorithms (For example see [2]). Thus in what follows we may assume the following.
G does not contain an induced cycle C, $4 \leq |C| \leq 8$ *and G does not contain a small AT as an induced subgraph, i.e. G does not contain small obstructions.*
This allows us to decompose G into smaller building blocks.

If G does not contain small AT's the it does not contain an induced C_k, $k \geq 4$ then either it is interval graph or it contains only two types of AT's so called *big AT* depicted in Figure 2.

Let $S_{a,b,c}$ denote an AT over the vertices a, b, c. $S_{a,b,c}$ with the vertex set $a, b, c, u, v_1, v_2, \ldots, v_p$ and the edge set

$$E(S_{a,b,c}) = \{av_1, v_1v_2, \ldots, v_{p-1}v_p, v_pb, uv_1, uv_2, \ldots, uv_p, uc\}$$

is called type 1 AT. $S_{a,b,c}$ with vertex set $a, b, c, u, w, v_1, v_2, \ldots, v_p$ and the edge set

$$E(S_{a,b,c}) = \{au, bw, cu, cw, av_1, bv_p, uv_1, wv_1\} \cup \{v_iv_{i+1}, uv_{i+1}, wv_{i+1} | 1 \leq i \leq p-1\}.$$

is called type 2 AT.

Definition 1.2 *We say AT,* $S_{a,b,c}$ *is* ripe *if the set of vertices outside the neighborhood of c and adjacent to some vertices in* $\{v_3, v_4, \ldots, v_{p-2}\}$ *induce an interval graph in G.*

The broad overview of the algorithm for k-interval deletion problem :

The Algorithm for interval vertex consists of two main steps.

Step 1) *G is a chordal graph.*

If G is not interval then according to our assumption it contains a big AT. We show that there exists a *ripe AT* in G. The ripe AT $S_{a,b,c}$ is the starting point of the algorithm. The algorithm starts with a ripe AT, $S_{a,b,c}$ and it proceeds as follows.

- Branch by deleting one of the vertices $\{a, b, u, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$
- Branch by deleting all the vertices in X, where X is a minimum set of vertices outside the neighborhood of c that separates v_6 from v_{p-5}.

For the correctness we show the following lemma.

Lemma 1.3 *Let G be a chordal graph without small AT's and let $S_{a,b,c}$ be a ripe AT. Let X be a minimum separator in $G \setminus N(c)$ that separates v_6 from v_{p-5} and X contains a v_j, $7 \leq j \leq p-6$. Then there is a minimum set of deleting vertices F such that $G \setminus F$ is an interval graph and at least one of the following holds:*

(i) F contains at least one vertex from

$$\{a, b, u, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$$

(iii) F contains all the vertices in X.

Step 2) *G is not chordal.* We start with the following definition.

Definition 1.4 *We say a shortest induced cycle C, $9 \leq |C|$ of G is* **clean** *if for every induced cycle $C_1 \neq C$ $(9 \leq |C_1|)$ in the the closed neighborhood of C, every vertex in C_1 is adjacent to at most three consecutive vertices in C and the closed neighborhood of C_1 contains C. We say C is* **ripe** *if it is clean and it does not have any AT in its neighborhood.*

We show the following statements.

1. If G is not chordal then there exists always a clean cycle C in G.

2. Consider clean cycle C that is not ripe. Let $S_{a,b,c}$ be a big AT such that $V(S_{a,b,c}) \subseteq N[C]$ ($N[C]$ is the closed neighborhood of C). Then $S_{a,b,c}$ lies in the union of the neighborhood of at most three consecutive vertices of the cycle C.

Step 2.1) Start with a clean cycle C. If C is not ripe then consider big AT $S_{a,b,c}$ in $N[C]$ and let u, v, w be three consecutive vertices of C such that $N[\{u, v, w\}]$ contains the vertex set $S_{a,b,c}$. Apply the algorithm for the chordal case on the subgraph of G induced by $N[\{u, v, w\}]$.

Step 2.2) Start with ripe cycle C. Find a minimum set X of vertices in the closed neighborhood of C whose deletion break all the cycles in the closed neighborhood of C. Set X is called a minimum *cycle-separator*. At this point the algorithm either deletes all the vertices in C, or it deletes all the vertices in X at once. We show that the choice of set Y is arbitrary. For the correctness of Step 2 we show the following lemma.

Lemma 1.5 *Let C be a ripe cycle and let X be a minimum cycle-separator in $N[C]$. Then there is a minimum set of deleting vertices F such that $G \setminus F$ is an interval graph and at least one of the following holds:*

(i) F contains all the vertices of the cycle C.

(ii) F contains all the vertices of X.

The overall complexity of the algorithm for k-interval deletion is $O(c^k n(m+n))$ where $c = \min\{18, k\}$. By using slightly more restricted definition for ripe AT we can get a better running time $O(12^k n(n+m))$.

A broad overview of the k-interval completion algorithm :

Suppose input graph G contains an induced cycle C of length at least 4. In order to obtain an interval graph we must add a set of at least $|C|-3$ edges into vertices of C, or equivalently we need to triangulate cycle C. It is not difficult to see that there are at most $O(4^{|C|-3})$ different ways of triangulating cycle C. Thus we branch on all different ways of triangulating cycle C, and after each of them the parameter k decreases by $|C|-3$.

For the sake of clarification and simplicity we just explain what we do when dealing with AT of type 1. The algorithm treats the type 2 AT very similar to the type 1.

We need to add at least one edge e to $S_{a,b,c}$ such that $S_{a,b,c} \cup \{e\}$ is no longer induces an AT in G. We add one of the edges cv_i, $1 \le i \le 6$ or one of the edges cv_i, $p-5 \le i \le p$ or we add one of the edges au, bu, ab. If we add edge ab then we need to triangulate the cycle with the vertices $a, v_1, \dots, v_p, b, a$. Now it remains to make a decision for adding one edge cv_j for some $7 \le j \le p-6$. We show that we can add the edge cv_j when v_j belongs to a minimum (v_6, v_{p-5})-separator outside the neighborhood of c. We show that the choice of v_j is arbitrary and it does not affect the global solution when v_j belongs to a minimum (v_6, v_{p-5})-separator outside the neighborhood of c. This allows us to get a single exponential FPT algorithm (see Figure 3). We prove the following lemma.

Lemma 1.6 *Let G be a chordal graph without small ATs and let $S_{a,b,c}$ be a minimum ripe AT with the path $P_{a,b} = a, v_1, v_2, \dots, v_p, b$. Let X be a minimum separator in $G \setminus N(c)$ that separates v_6 from v_{p-5} and it contains a vertex v_i, $7 \le i \le p-6$. Then there is a minimum set of edges F such that $G \cup F$ is an interval graph and at least one of the following holds:*

(i) F contains at least one edge from

$$\{bu, au, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

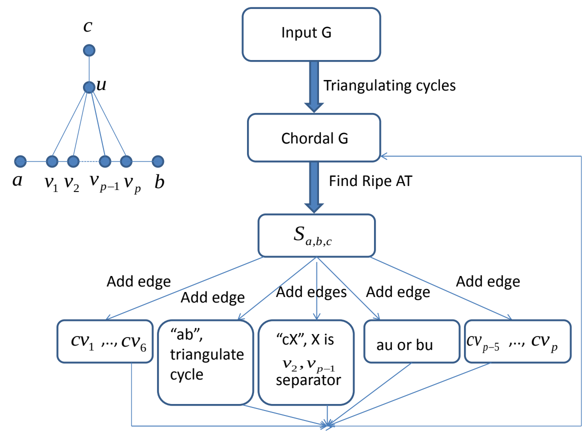

Figure 3: Edge Addition Flow

(ii) *F contains all the edges* av_i, $2 \leq i \leq p$ *and* ab.

(iii) *F contains all the edges* bv_i, $1 \leq i \leq p-1$ *and* ab.

(iv) *F contains all edges* $E_X = \{cx | x \in X\}$.

There are several technical and structural lemmas required to obtain a minimum ripe AT $S_{a,b,c}$ and to obtain a decomposition of G into building blocks when G is free from small interval graph obstructions. However the main idea is what explained. Overall the running time of the algorithm is $O(c^k n(n+m))$, $c = \min\{17, k\}$. By using slightly more restricted definition for ripe AT we can get a better running time $O(11^k n(n+m))$.

The paper is organized as follows. Section 2 is for terminology and notations. In Section 3 we investigate the structure of a chordal graph G which does not contain small ATs as induced subgraphs. We start with a minimum AT, and then we obtain a minimum ripe AT $S_{a,b,c}$. Next we consider the interaction (vertex intersection) of another minimum AT, $S_{x,y,z}$ with $S_{a,b,c}$. The $S_{a,b,c}$ and $S_{x,y,z}$ interact in a very particular way. In Section 4 we consider k-interval deletion problem. In Subsection 4.1 we consider the case when G is chordal and does not contain neither small ATs as induced subgraph. If G is chordal then the results in Section 3 with regard to the vertex interaction of $S_{a,b,c}$ and $S_{x,y,z}$ enable us to reduce the number of branches in a search tree into a constant number and hence we obtain an efficient FPT algorithm. In Subsection 4.2 we deal with the non-chordal case. We start with a shortest cycle C and we show that any other induced cycle (of length more than 8) in $N[C]$ interact with C in a special way due to absence of the small obstructions. The interaction between cycle C and AT $S_{a,b,c}$ is investigated and we show that either $S_{a,b,c}$ lies in the neighborhood of at most three consecutive vertices of C or the entire path $P_{a,b}$ of $S_{a,b,c}$ lies outside the neighborhood of C. Finally the main algorithm is presented at Section 4.3 and its correctness is proved. In Section 5 we consider the Interval completion problem. In Subsection 5.1 we further investigate the edge interaction of $S_{a,b,c}$ and $S_{x,y,z}$, i.e. when $S_{a,b,c}$ and $S_{x,y,z}$ have an edge in common. The edge interaction occurs in a special way and we make a use of it to get a single exponential FPT for k-interval completion

problem. In Subsection 5.2 we present the main algorithm for interval completion problem and we prove its correctness.

2 Terminology and Notations

We consider simple, finite, and undirected graphs. For a graph G, $V(G)$ is the *vertex set* of G and $E(G)$ is the *edge set* of G. For every edge $uv \in E(G)$, vertices u and v are *adjacent* or *neighbors*. The *neighborhood* of a vertex u in G is $N_G(u) = \{v \mid uv \in E(G)\}$, and the *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. When the context will be clear we will omit the subscript. A set $X \subseteq V$ is called *clique* of G if the vertices in X are pairwise adjacent. A *maximal* clique is a clique that is not a proper subset of any other clique. For $U \subseteq V$, the *subgraph of G induced by U* is denoted by $G[U]$ and it is the graph with vertex set U and edge set equal to the set of edges $uv \in E$ with $u, v \in U$. For every $U \subseteq V$, $G' = G[U]$ is an *induced subgraph* of G. By $G \setminus X$ for $X \subseteq V$, we denote the graph $G[V \setminus X]$. For two disjoint subsets X, Y of $V(G)$, $S \subset G - (X \cup Y)$ is a (X, Y)-separator if there is no path from any vertex of X to any vertex in Y in $G \setminus S$. A graph is *chordal* if each of its cycles of four or more vertices has a chord, which is an edge joining two vertices that are not adjacent in the cycle.

3 Structure when G is chordal and there are no small AT's

In this section we assume that G is chordal and it does not contain small AT's (See Figure 1). By the results of Lekkerkerker and Boland [18], every other possible minimal AT in G is one of two graphs depicted in Figure 2.

Let $S_{a,b,c}$ denote an AT with the vertices a, b, c, such that the path between a, c and the path between b, c are of length 2 and the path between a, b has length at least 7. Vertex c is called a *shallow vertex* .

Definition 3.1 *We say AT $S_{a,b,c}$ is of type 1 if $S_{a,b,c}$ has the vertex set $\{a, b, c, u, v_1, v_2, ..., v_p\}$ and the edge set*

$$\{av_1, cu, bv_p, uv_1\} \cup \{v_iv_{i+1}, uv_{i+1} | 1 \leq i \leq p - 1\}.$$

Vertex u is called a center vertex. *We set $v_0 = a$ and $v_{p+1} = b$.*

Definition 3.2 *We say AT $S_{a,b,c}$ is of type 2 if $S_{a,b,c}$ has the vertex set $\{a, b, c, u, w, v_1, v_2, ..., v_p\}$ and the edge set*

$$\{au, bw, cu, cw, av_1, bv_p, uv_1, wv_1\} \cup \{v_iv_{i+1}, uv_{i+1}, wv_{i+1} | 1 \leq i \leq p - 1\}.$$

The vertices u, w are called central vertices. *We set $v_0 = a$ and $v_{p+1} = b$.*

Let G' be an induced subgraph of G, and let $S_{a,b,c}$ be an AT in G'. We say $S_{a,b,c}$ is *minimum* if among all the AT, $S_{a',b',c'}$ in G' the path between a, b in $S_{a,b,c}$ has the minimum number of vertices and if there is a choice we assume that $S_{a,b,c}$ is of type 1. We denotes the path $a, v_1, v_2, \ldots, v_p, b$ by $P_{a,b}$.

Definition 3.3 *We say a vertex x is a dominating vertex for $S_{a,b,c}$ if x is adjacent to all the vertices $v_1, v_2, v_3, \ldots, v_{p-1}, v_p$.*

In the rest of this paper the set of dominating vertices for $S_{a,b,c}$ is denoted by $D(a, b, c)$. The following lemma shows the relationship of minimum $S_{a,b,c}$ with the other vertices of G.

Lemma 3.4 *Let G be a chordal graph without small ATs. Let $S_{a,b,c}$ be a minimum AT with a path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let x be a vertex in $G \setminus S_{a,b,c}$. Then the following hold.*

(1) If cx is an edge of G then ux is an edge of G when $S_{a,b,c}$ is of type 1, and xu, xw are edges of G when $S_{a,b,c}$ is of type 2.

(2) If $S_{a,b,c}$ is of type 1 and x is adjacent to v_j, for some $2 \leq j \leq p-1$, then x is adjacent to u.

(3) If $S_{a,b,c}$ is of type 2 and x is adjacent to v_j, for some $1 \leq j \leq p$, then x is adjacent to both u and w.

(4) Let vertex x be adjacent to c. If x is adjacent to v_i, for some $0 \leq i \leq p+1$, then x is a dominating vertex for $S_{a,b,c}$.

(5) Every vertex $x \in G \setminus N(c)$ is adjacent to at most three vertices of the path $a, v_1, \ldots, v_p, b$. Moreover, the neighbors of x are consecutive vertices in the path $a, v_1, \ldots, v_p, b$.

(6) If $x \in G \setminus N(c)$ is adjacent to v_i, $3 \leq i \leq p-2$, then every vertex $y \in G \setminus N(c)$ adjacent to x, is also adjacent to at least one of the vertices v_j, $i-2 \leq j \leq i+2$.

(7) If x is adjacent to some v_i, $2 \leq i \leq p-1$, then x is adjacent to every dominating vertex y.

(8) If $x \in G \setminus N(c)$ is adjacent to some v_i, $2 \leq i \leq p-1$, and x is adjacent to some $y \in N(c)$, then y is a dominating vertex .

Proof: (1). Let us first suppose that $S_{a,b,c}$ is of type 1. If xu is not an edge of G, then x should be adjacent to at least one of the vertices v_1, v_p, a, and b, because otherwise vertices x, c, u, v_1, v_p, a, b induce a small AT in G. If xa is an edge, then because G is chordal, the cycle induced by $\{x, c, u, v_1, a\}$ should have chord xv_1. Similarly, if $\{x, b\}$ are adjacent, so should be $\{x, v_p\}$. But neither xv_1, nor xv_p can form an edge of G because otherwise we obtain an induced 4-cycle x, c, u, v_1 or x, c, v_p, b in chordal graph G.

Now suppose that $S_{a,b,c}$ is of type 2. Targeting towards a contradiction, let us assume that xw is not an edge. Then xb is not an edge because otherwise x, c, w, b would induce C_4 in G. Furthermore, xv_1 is not an edge because otherwise C_4 is induced by vertices x, c, w, and v_1. We also note that xa is not an edge as otherwise x, a, v_1, w, c would induce C_5 in G. Thus if x is not adjacent to w, then x cannot be adjacent to a, b and v_1. But then set $\{x, c, u, w, v_1, v_p, a, b\}$, even when x and u are adjacent, induces a small AT in G, which is a contradiction. Similar argument implies that xu is an edge.

(2). If xu is not an edge then by (1), vertices x and c are not adjacent. Then vertex x has at most three neighbors among the vertices of path $P_{a,b}$. This is because otherwise, there will be a shorter (a, b)-path in G passing through x and avoiding the closed neighborhood of c. But then

vertices of this paths together with u and c induce an AT $S'_{a,b,c}$ of size smaller than the size of $S_{a,b,c}$. This is a contradiction to the choice of $S_{a,b,c}$. Thus x has at most three neighbors in $P_{a,b}$. Let v_i, $i \leq j$, be the leftmost neighbor of x in $P_{a,b}$, and v_k, $k \leq j$, be the rightmost neighbor. We observe that $k - i \leq 2$, because otherwise we obtain an induced cycle of length at least four in G. Because G has no small ATs and thus $n \geq 7$, we have that either $i \geq 2$, and in this case vertices $a, v_1, v_2, ..., v_i, x, c, u$ induce a smaller AT than $S_{a,b,c}$, or $k \leq p-1$, and then $x, v_k, v_{k+1}, ..., v_p, b, u, c$ form a smaller AT.

(3). The proof here is similar to the proof of (2).

(4). We prove the statement when $S_{a,b,c}$ is of type 1. The argument for when $S_{a,b,c}$ is of type 2 is similar. By (1), xu is an edge. If x is adjacent to v_i for some $0 \leq i \leq p-1$, then xv_{i+2} is also an edge of G as otherwise the vertices $a, v_1, \ldots, v_{i+2}, c, x$ induce a smaller AT $S_{a,v_{i+2},c}$. In this case we note that xv_{i+1} is also an edge because vertices x, v_i, v_{i+1}, v_{i+2} would induce C_4 otherwise. Similarly if x is adjacent to v_j, $2 \leq j \leq p+1$, then xv_{j-2} is an edge as otherwise the vertices $b, v_p, v_{p-1}, \ldots, v_{i-2}, c, x$ induce smaller AT $S_{a,v_{i-2},c}$. In this case we note that xv_{i-1} is also an edge as otherwise there would be an induced C_4 on x, v_i, v_{i-1}, v_{i-2}. By applying these arguments inductively, we obtain that x is adjacent to every v_i, for $2 \leq j \leq p-1$. Now if none from the pairs xv_1, xv_p is an edge, then $v_1, v_2, \ldots, v_p, x, c$ induce smaller AT $S_{v_1,v_p,c}$, a contradiction. Therefore we may assume that x should be adjacent either to v_1, or to v_p. Let us assume, without loss of generality, that x is adjacent to v_1. Now if xv_p is not an edge, then $a, v_1, v_2, \ldots, v_{p-1}, c, x$ is a smaller AT when ax is not an edge. We conclude that if xv_p is not an edge, then xa, xv_1 are edges of G. However $c, x, u, a, v_1, v_2, \ldots, v_p$ induce an AT $S_{a,v_p,c}$ of type 2 and the path between a, v_p is shorter the path between a, b in $S_{a,b,c}$, this is a contradiction. Therefore xv_p is an edge.

(5). If there was a vertex $x \in G \setminus N(c)$ adjacent to more than three vertices in the path $P_{a,b}$ then there is a shorter path between a, b using vertex x avoiding neighborhood of c. Thus we construct a smaller AT. The neighbors are consecutive vertices of the path because otherwise we obtain an induced cycle of length at least four.

(6). If y is adjacent to none of the vertices $v_{i-2}, v_{i-1}, \ldots, v_{i+2}$, then vertices $y, x, v_i, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$ induce a smaller AT unless x is adjacent to v_{i-2} or v_{i+2}. Suppose that x is adjacent to v_{i-2}. Now by (6), x is adjacent to v_{i-1}. By (5), x cannot be adjacent to more than 3 vertices of the path $v_1, \ldots, v_p$, and thus x is not adjacent to v_{i-3} and v_{i+1}. Vertex y is not adjacent to v_{i-3} because vertices v_{i-3}, v_{i-2}, x, y do not induce a cycle. In this case vertices $v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_{i+1}, x, y$ induce a small AT.

(7). If x is adjacent to c then by (4), x is a dominating vertex and hence x is adjacent to y as otherwise x, y, v_1, v_3 induces a C_4. So we may assume that $x \notin N(c)$. By (5), y should be adjacent to c. In this case, if x is not adjacent to y, then either $c, u, x, y, v_i, v_{i+1}, \ldots, v_p, b$, or $a, v_1, v_2, \ldots, v_i, y, x, u, c$ induce a smaller AT.

(8). If y is adjacent to at least one vertex v_i for some $0 \leq i \leq p+1$, then by (4) y is a dominating vertex for $S_{a,b,c}$. Let us assume that y is non-adjacent to all vertices v_i, $0 \leq i \leq p+1$. Now $S_{a,b,y}$ has exactly the same number of vertices as $S_{a,b,c}$, and thus is also a minimum AT. By applying item (4) for $S_{a,b,y}$ we conclude that x is a dominating vertex for $S_{a,b,y}$ and hence x is adjacent to more than three vertices in the path $P_{a,b} = a, v_1, \ldots, v_p, b$. This is a contradiction to (5) because by assumption $x \in G \setminus N(c)$.

$\diamond$

The following Lemma follows from item (4) of Lemma 3.4.

Lemma 3.5 *Let $S_{a,b,c}$ be a minimum AT with a path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (central vertices u, w). Let Q be a chordless path from c to some v_i, $0 \leq i \leq p+1$. Then the second vertex of Q is a dominating vertex for $S_{a,b,c}$. Moreover if $1 \leq i \leq p$ then the length of Q is 2.*

Proof: Let $Q = c, c_1, c_2, \ldots, c_r, v_i$ be a chordless path from c to v_i, $0 \leq i \leq p+1$. Suppose c_1 is adjacent to some vertex v_j, $0 \leq j \leq p+1$. Then by Lemma 3.4(4), c_1 is a dominating vertex for $S_{a,b,c}$ and hence $c_1 v_j$ is an edge for every $1 \leq j \leq p$. If $j \neq 0, p+1$ then $c_1 v_i$ is an edge and hence $r = 1$ and lemma is proved. Thus we assume that c_1 is not adjacent to any v_j, $0 \leq j \leq p+1$. By Lemma 3.4 (1), uc_1 is an edge. This implies that S_{a,b,c_1} is an AT with the same number of vertices as $S_{a,b,c}$. Now by applying the same argument for S_{a,b,c_1} we conclude that c_2 is a dominating vertex for S_{a,b,c_1}. However by item (6) of Lemma 3.4 for $S_{a,b,c}$, c_2 is a dominating vertex for $S_{a,b,c}$ and hence by item (5) of Lemma 3.4 we conclude that c_2 is adjacent to c. This is a contradiction to Q being a chordless path.

$\diamond$

Let G be a chordal graph without small ATs. Let $S_{a,b,c}$ be a minimum AT in G. Then by item (5) of Lemma 3.4, every vertex x of $G \setminus N(c)$ has at most three neighbors in the $P_{a,b}$ path, and moreover, these neighbors should be consecutive vertices of this path. Note that we assume $v_0 = a$ and $v_{p+1} = b$. We introduce the following notations. We define the following subsets of $G \setminus N(c)$

- S_i vertices adjacent to v_i and not adjacent to any other v_j, $j \neq i$, $1 \leq i \leq p$;
- D_i vertices adjacent to v_i, v_{i+1} and not adjacent to any other v_j, $j \neq i, i+1$, $0 \leq i \leq p$;
- T_i vertices adjacent to v_i, v_{i+1}, v_{i+2}, $0 \leq i \leq p-1$.

The following corollary is obtained from Lemma 3.4 (1,7,8).

Corollary 3.6 *Let $S_{a,b,c}$ be a minimum AT. Then the vertices in $D(a,b,c)$ form a clique. Every vertex adjacent to c is also adjacent to every dominating vertex. Moreover every vertex in $D(a,b,c)$ is also adjacent to c.*

Definition 3.7 *For minimum AT, $S_{a,b,c}$ let $B[a,b]$ be the set of vertices in $D_0 \cup T_0 \cup D_1 \cup S_1 \cup \{v_1\} \cup S_2$ and $E[a,b]$ be the set of the vertices in $S_{p-1} \cup D_{p-1} \cup T_{p-1} \cup D_p \cup S_p \cup \{v_p\}$.*

Definition 3.8 *For minimum AT, $S_{a,b,c}$ let $G[a,b,c] = G[\{x | x \in N[v_i] \setminus N(c); 3 \leq i \leq p-2\}]$.*

Since every vertex in $G[a,b,c]$ is adjacent to some v_i, $3 \leq i \leq p-2$ by Lemma 3.4(7) we have the following.

Corollary 3.9 *Every vertex in $G[a,b,c]$ is adjacent to every vertex in $D(a,b,c)$.*

Lemma 3.10 *Let x be a vertex adjacent to some vertex in $G[a,b,c]$. Then x is adjacent to every vertex in $D(a,b,c)$.*

Proof: If $x \in N(c)$ then by Corollary 3.6 the Lemma holds. Therefore we may assume that $x \notin N(c)$. Let xx' be an edge of G for some $x' \in G[a,b,c]$. By definition of $G[a,b,c]$, x' is adjacent to some v_i, $3 \leq i \leq p-2$. By Lemma 3.4 (6), x is adjacent to some v_j, $i-2 \leq j \leq i+2$. If x is adjacent to one of the v_{i-1}, v_i, v_{i+1} then by Lemma 3.4(7) x is adjacent to every vertex in $D(a,b,c)$.
Therefore w.l.o.g assume that x is adjacent to v_{i-2} and not adjacent to any of v_{i-1}, v_i. Now we observe that x' is adjacent to v_{i-2}, v_{i-1}, v_i as otherwise we obtain an induced C_4 or induced C_5 with the vertices $v_{i-2}, v_{i-1}, v_i, x', x$. Now by replacing v_{i-1} with x' we obtain a minimum AT $(S_{a,b,c})'$ with the same number of vertices as $S_{a,b,c}$, and path $P'_{a,b} = a, v_1, \dots, v_{i-2}, x'$, $v_i, \dots, v_p, b$. Note that $2 \leq i-1 \leq p-1$. Thus the set $D(a,b,c)$ is also the set of dominating vertices for $(S_{a,b,c})'$. Now because xx' is an edge Lemma 3.4(7) for $(S_{a,b,c})'$ implies that x is adjacent to every vertex in $D(a,b,c)$.

Lemma 3.11 *Let $S_{a,b,c}$ be a minimum AT. Then $D(a,b,c) \cup B[a,b] \cup E[a,b]$ separates $G[a,b,c]$ from the rest of the graph.*

Proof: We need to show that if $x \in G \setminus (\{v_1, v_p\} \cup D_0 \cup S_1 \cup D_1 \cup T_0 \cup S_p \cup D_{p-1} \cup T_{p-1} \cup D_p \cup D(a,b,c) \cup V(G[a,b,c]))$ then there is no edge from x to $y \in G[a,b,c]$. For contradiction suppose xy is an edge. We also note that $x \notin N(c)$ as otherwise by Lemma 3.5, x is a dominating vertex for $S_{a,b,c}$ and we get a contradiction. By definition of $G[a,b,c]$, y is adjacent to v_i, $3 \leq i \leq p-2$. First suppose $y \in \{v_2, v_{p-1}\}$. Now x is adjacent to v_2 or x is adjacent to v_{p-1}. Since x is adjacent to at most three consecutive vertices on the path $P_{a,b}$, x lies in $\{v_1, v_p\} \cup S_2 \cup T_0 \cup S_p \cup D_{p-1} \cup T_{p-1} \cup D_p$. This implies that $x \in B[a,b] \cup E[a,b]$. We continue by assuming that $y \in V(G[a,b,c]) \setminus \{v_2, v_{p-1}\}$. Now we apply Lemma 3.4(6) for y and we conclude that x is adjacent to one of the vertices $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}$. Therefore x is in $N[v_r]$, $1 \leq r \leq p$. We observe that $r \in \{1, 2, p-1, p\}$ as otherwise by definition x is in $G[a,b,c]$. Therefore $x \in B[a,b] \cup E[a,b]$. $\diamond$

The following Lemma is obtained by applying similar argument in Lemma 3.5.

Lemma 3.12 *Let $S_{a,b,c}$ be a minimum AT with a path $P_{a,b} = a, v_1, v_2, \dots, v_p, b$ and center vertex u (u, w). Then every chordless path from c to $d \in G[a,b,c]$ has length 2 and the intermediate vertex of this path is a dominating vertex for $S_{a,b,c}$.*

Lemma 3.13 *Let $S_{x,y,z}$ be a minimum AT in $G[a,b,c]$ with a path $P_{x,y} = x, w_1, w_2, \dots, w_q, y$ ($x = w_0$, $y = w_{q+1}$) and center vertex u' (central vertices u', w' if of type 2). Then there exists $2 \leq i \leq p-1$, such that v_i is a dominating vertex for $S_{x,y,z}$.*

Proof: If $u' \in \{v_{i-1}, v_i, v_{i+1}\}$ then $u' = v_j \in \{v_{i-1}, v_i, v_{i+1}\}$ is a dominating vertex for $S_{x,y,z}$. Note that by definition of $G[a,b,c]$ we have $2 \leq j \leq p-1$. Therefore we may assume that u' is not on the path $P_{a,b}$.

We first show that $v_i \neq z$. For contradiction suppose $z = v_i$. Observe that the conditions of the Lemma 3.4(1) are applied for $S_{x,y,z}$ and hence v_{i-1} is adjacent to u' and v_{i+1} is adjacent to u'. First suppose v_{i+1} is not adjacent to any vertex w_j, $0 \leq j \leq q+1$. By replacing v_i with

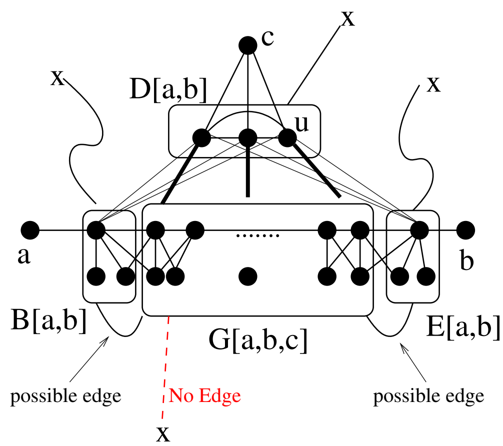

Figure 4: G[a,b,c] and outside

v_{i+1} we get a minimum AT, $S_{x,y,v_{i+1}}$ with the same number of vertices as S_{x,y,v_i}, and hence by Lemma 3.4(1), v_{i+2} must be adjacent to u'. This implies that u' is adjacent to more than three vertices of the path $P_{a,b}$. Since $u' \notin N(c)$, we get a contradiction by Lemma 3.4(5). Therefore v_{i+1} must be adjacent to some w_j. Similarly we conclude that v_{i-1} must be adjacent to some $w_{j'}$, $0 \le j' \le q+1$. By applying the item (4) of Lemma 3.4 for S_{x,y,v_i}, v_{i-1} is a dominating vertex for S_{x,y,v_i} and similarly v_{i+1} is also a dominating vertex for $S_{x,y,v_{i+1}}$. But this is a contradiction because by Corollary 3.6 $v_{i-1}v_{i+1}$ is an edge. Therefore we have the following fact.

(f) For every minimum AT, $S_{x',y',z'} \subseteq G[a,b,c]$ we have $z' \neq v_i$, $2 \le i \le p-1$.

Now suppose $z \in S_i \cup D_i \cup T_i$ and $v_i z$ is an edge of G. Note that $v_i u'$ is also an edge by Lemma 3.4(1). Now if v_i is not adjacent to any vertex of the path $P_{x,y}$ then S_{x,y,v_i} is also a minimum AT with the same number of vertices as $S_{x,y,z}$ and we get a contradiction by (f). Thus we conclude that v_i is adjacent to some vertex w_j, $0 \le j \le q+1$ and hence by Lemma 3.4(4), v_i is a dominating vertex for $S_{x,y,z}$.

$\diamond$

Definition 3.14 *We say an AT, $S_{a,b,c}$ is* ripe *if there is no AT in $G[a,b,c]$ i.e. $G[a,b,c]$ is an interval graph.*

Remark : Note that a ripe AT may not be necessary a minimum AT. But for the purpose of the algorithm we often use a minimum ripe AT and by that we mean an AT which is ripe and it is minimum among all the ripe AT' in a subgraph of G.

Looking for a minimum ripe AT, starting with a minimum AT S_{a_0,b_0,c_0}.

We start with minimum AT, S_{a_0,b_0,c_0} and put an arc from S_{a_0,b_0,c_0} to minimum AT, S_{a_1,b_1,c_1} in $G[a_0,b_0,c_0]$. Note that according to Lemma 3.13 there is a vertex v_i (on the path P_{a_0,b_0}) that is a dominating vertex for S_{a_1,b_1,c_1}. We say S_{a_0,b_0,c_0} dominates S_{a_1,b_1,c_1} at v_i. Now we define the sets S_i^1, D_i^1, T_i^1 with respect to the vertices on the path $P_{a_1,b_1} = a_1, w_1, w_2, \ldots, w_q, b_q$ in the same way we defined them for S_{a_0,b_0,c_0}. We continue for other AT's dominated by S_{a_1,b_1,c_1}. See the Algorithm 1 for more details.

Algorithm 1 Looking for a minimum ripe AT

1. Start with an arbitrary minimum AT, S_{a_0,b_0,c_0}, and set $i = 0$, $G_0 = G$.
2. Define $G_i[a_i, b_i, c_i]$ in $G_i \setminus N(c_i)$ (see definition 3.8) and set $G_{i+1} = G_i[a_i, b_i, c_i]$.
3. If there is no AT in G_{i+1}, report S_{a_i,b_i,c_i} as a ripe AT and exit.
4. If $i > k$ then report NO solution and exit.
5. Let $S_{a_{i+1},b_{i+1},c_{i+1}}$ be a minimum AT, in G_{i+1}
6. put an arc from S_{a_i,b_i,c_i} to $S_{a_{i+1},b_{i+1},c_{i+1}}$,
7. increase i by one and go to (2).

Lemma 3.15 *The Algorithm 1 reports a ripe AT and terminates after at most* $2k$ *steps.*

Proof: Suppose $S_{a,b,c}$ dominates S_{a^1,b^1,b^1} at some vertex v_i. Then by Lemma 3.13, v_i is a dominating vertex for S_{a^1,b^1,c^1}. If S_{a^1,b^1,c^1} also dominates S_{a^2,b^2,c^2} at some vertex w_j on the path P_{a^1,b^1} then by Lemma 3.4 item (2) or (3) the vertices of S_{a^2,b^2,c^2} are all adjacent to v_i (since v_i is a dominating vertex for $S_{a,b,c}$). Now it is easy to see that there is no arc from S_{a^r,b^r,c^r} to $S_{a,b,c}$, as otherwise the vertices on the path $S_{a,b,c}$ must be all adjacent to some vertex in the neighborhood of v_i which is not possible. Therefore there is no arc from an AT at step i in the Algorithm 1 to an AT at step $j < i$. Thus the algorithm reports a ripe AT after at most k steps. Note that the number of AT's found in the Algorithm 1 can not be more than k. $\diamond$

3.1 AT and AT interaction

Remark : In the following three Lemmas we consider the interaction of a minimum AT, $S_{x,y,z}$ with a minimum ripe AT, $S_{a,b,c}$. There are only four possible interaction configurations for these two ATs. In two of these configurations the central vertex (vertices) of $S_{x,y,z}$ lie in dominating set of $S_{a,b,c}$. In two of these configurations the path $P_{x,y}$ has no intersection with $G[a, b, c]$ and in one situation every vertex in $P_{a,b}$ has a neighbor in $P_{x,y}$. In two situations if $V(S_{x,y,z}) \cap N[v_i] = T$, for some $7 \le i \le p - 6$ then $T = \{x\}$ or $T = \{y\}$. See the Figures 4,5,6.

Lemma 3.16 *Let* $S_{a,b,c}$ *be a minimum ripe AT. Let* $S_{x,y,z}$ *be a minimum AT with a path* $P_{x,y} = x, w_1, \ldots, w_q, y$*, and center vertex (central vertices)* u' (u', w') *such that* $u' \in D(a, b, c)$ $(u', w' \in D(a, b, c)$ *if of type 2) and* $V(S_{x,y,z}) \cap G[a, b, c] \neq \emptyset$*. Then one of the following happens:*

1. $P_{x,y} \cap B[a, b] \neq \emptyset$ *and* $P_{x,y} \cap E[a, b] \neq \emptyset$ *and every* v_i, $1 \le i \le p$ *has a neighbor in* $P_{x,y}$ *(See the Figure 3.1).*
2. $z \in G[a, b, c]$ *and* $P_{x,y} \cap G[a, b, c] = \emptyset$ *and for every vertex* $z' \in G[a, b, c]$, $S_{x,y,z'}$ *is an AT with the same path* $x, w_1, w_2, \ldots, w_q, y$ *(See the Figure 3.1).*

Proof: By Corollary 3.9 every dominating vertex is adjacent to every vertex in $G[a, b, c]$. Therefore none of the x, y is in $G[a, b, c]$ as otherwise xu' $(xu', xw'$ when $S_{a,b,c}$ is of type 2) or yu' $(yu', yw'$ when $S_{a,b,c}$ is of type 2) is an edge. Moreover by Corollary 3.6 $x, y \notin D(a, b, c)$.

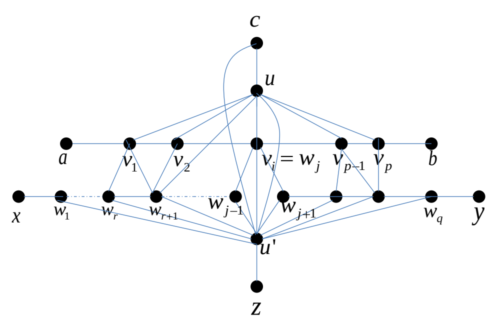

Figure 5: $u' \in D(a,b,c)$ and $P_{x,y} \cap B[a,b] \neq \emptyset$, $P_{x,y} \cap E[a,b] \neq \emptyset$

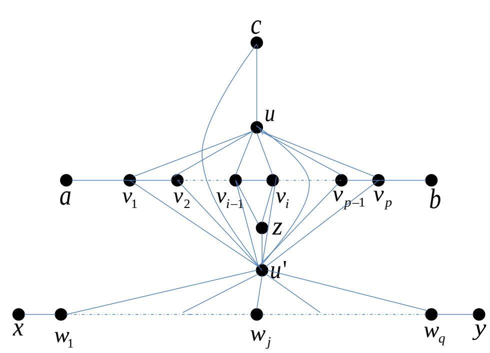

Figure 6: $z \in G[a,b,c]$ and $P_{x,y} \cap G[a,b,c] = \emptyset$

Now since $S_{x,y,z}$ has intersection with $G[a,b,c]$ we have two cases:

Case 1. $P_{x,y} \cap G[a,b,c] \neq \emptyset$. There exists some w_j, such that $w_j \in G[a,b,c]$. We show that $2 \leq j \leq q-1$. Otherwise w.l.o.g assume that $w_1 \in G[a,b,c]$. Since xw_1 is an edge by Lemma 3.10 x is adjacent to every vertex in $D(a,b,c)$ and in particular x is adjacent to u' (u', w') and hence we get a contradiction.

We continue by the assumption that $w_j \in G[a,b,c]$ and $2 \leq j \leq p-2$. By definition of $G[a,b,c]$, w_j is adjacent to some vertex v_i, $3 \leq i \leq p-2$.

We first show that $P_{x,y} \cap D(a,b,c) = \emptyset$. For contradiction suppose $w_t \in D(a,b,c)$, $1 \leq t \leq q$. Now by Corollary 3.9 w_t is adjacent to w_j and hence $t = j+1$ or $t = j-1$. W.l.o.g assume that $t = j+1$. Since $w_{j-1}w_j$ is an edge of G and w_{j+1} is a dominating vertex for $S_{a,b,c}$, by Lemma 3.10, $w_{j-1}w_{j+1}$ is an edge of G, a contradiction. Therefore $P_{x,y} \cap D(a,b,c) = \emptyset$.

Since $x, y \notin G[a,b,c]$ and no vertex of $P_{x,y}$ is in $D(a,b,c)$, by Lemma 3 we conclude $B[a,b] \cap P_{x,y} \neq \emptyset$ or $E[a,b] \cap P_{x,y} \neq \emptyset$.

Observation 1. If for some v_i, $N[v_{i+1}] \cap P_{x,y} \neq \emptyset$ and $N[v_{i-1}] \cap P_{x,y} \neq \emptyset$ then $N[v_i] \cap P_{x,y} \neq \emptyset$. Otherwise we get an induced cycle of length at least four with the vertices v_{i-1}, v_i, v_{i+1} and part of $P_{x,y}$ from $N[v_{i+1}]$ to $N[v_{i-1}]$.

Now by Observation 1 if $B[a,b] \cap P_{x,y} \neq \emptyset$ and $E[a,b] \cap P_{x,y} \neq \emptyset$ we conclude (1). Therefore w.o.l.g assume that $B[a,b] \cap P_{x,y} = \emptyset$ and $E[a,b] \cap P_{x,y} \neq \emptyset$.

By Observation 1 and because $B[a,b] \cap P_{x,y} = \emptyset$ we conclude that there exists a maximum number $3 \leq r \leq p-2$ such that $N[v_r] \cap P_{x,y} \neq \emptyset$ and for every $1 \leq \ell \leq r-1$, $N[v_\ell] \cap P_{x,y} = \emptyset$. Now let i' be the first index such that $w_{i'}$ is in $N[v_r]$ and j' is the last index such that $w_{j'}$ is in $N[v_r]$. Recall that $2 \leq i', j' \leq q-1$ (Note that j' could be the same as i'). However $v_{r-2}, v_{r-1}, v_r, w_{i'}, w_{i'-1}, w_{i'-2}, w_{j'}, w_{j'+1}, w_{j'+2}$ induce a small AT.

Case 2. $P_{x,y} \cap G[a,b,c] = \emptyset$. Since $G[a,b,c] \cap V(S_{x,y,z}) \neq \emptyset$, $z \in G[a,b,c]$. By definition of $G[a,b,c]$; z is adjacent to some vertex v_i, $3 \leq i \leq p-2$. We show that v_i is not adjacent to any vertex w_j, $0 \leq j \leq q+1$. Otherwise by applying Lemma 3.4(4) for $S_{x,y,z}$; v_i is a dominating vertex for $S_{x,y,z}$ and now v_iw_1 implies that w_1 is in $G[a,b,c]$ which is a contradiction. (Note that w_j is not in $D(a,b,c)$ since zw_1 is not an edge). Therefore S_{x,y,v_i} is minimum AT and has the same number of vertices as $S_{x,y,z}$ and the same path $P_{x,y}$.

Now by repeating the same argument for S_{x,y,v_j} starting from $j = i$ and vertex v_{j+1} and vertex v_{j-1} (if they are in the range, v_3 and v_{p-2}) we conclude that

(f) For every $3 \leq j \leq p-2$, S_{x,y,v_j} is a minimum AT and the same number of vertices as $S_{x,y,z}$ and the same path $P_{x,y}$.

Now by applying similar argument for $z' \in G[a,b,c] \cap N(v_i)$; $3 \leq i \leq p-2$. We conclude that z' is not adjacent to some vertex w_j, $0 \leq j \leq q+1$. Otherwise by applying Lemma 3.4(4) for S_{x,y,v_i}, z' is a dominating vertex for S_{x,y,v_i}. We note that since S_{x,y,v_j} is a minimum AT with the same path $P_{x,y}$, z' is also a dominating vertex for S_{x,y,v_j} and hence by Corollary 3.6 z' is adjacent to v_j. This implies that z' is adjacent to every vertex v_r, $3 \leq r \leq p-2$, contradiction to $z' \in G[a,b,c]$. Therefore z' is not adjacent to any vertex on the path $P_{x,y}$ and hence $S_{x,y,z'}$ is

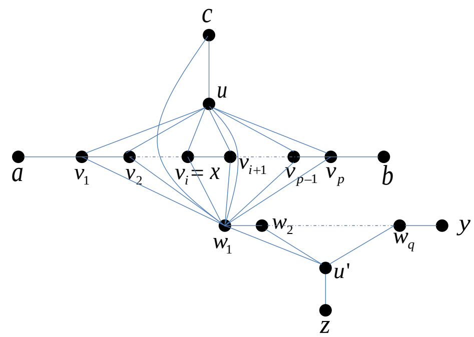

Figure 7: $x \in N[v_i]$, $w_1 \in D(a,b,c)$

a minimum AT with the same number of vertices as $S_{x,y,z}$. The proof of this case is complete.

$\diamond$

Lemma 3.17 *Let x_1, x_2, x_3 be three vertices in $G \setminus N(c)$ such that $v_i x_1, x_1 x_2, x_2 x_3$; $7 \leq i \leq p-6$ are edges of G. Then $x_3 \in N[v_j]$, $i-3 \leq j \leq i+3$*

Proof: By Lemma 3.4(6), x_2 is adjacent to one of the v_j, $i-2 \leq j \leq j+2$. If x_2 is adjacent to one of the v_{i-1}, v_i, v_{i+1} then by applying Lemma 3.4(6) for x_2, x_3 we conclude that x_3 is adjacent to some v_r, $i-3 \leq r \leq i+3$ and we are done. Thus w.l.o.g we may assume that x_2 is adjacent to v_{i-2} and not adjacent to any of v_{i-1}, v_i. Now x_1 is adjacent to v_{i-2}, v_{i-1}, v_i as otherwise we get an induced cycle of length 4 or 5 with the vertices $x_2, x_1, v_{i-1}, v_i, v_{i-2}$. Because $x_2 v_{i-2}, x_2 x_3$ are edges of G and $2 \leq i-2 \leq p-2$ by Lemma 3.4 (6) we conclude that x_3 is adjacent to v_{i-4} and not adjacent to any of v_{i-3}, v_{i-2} otherwise the Corollary holds. Now in this case x_2 must be adjacent to v_{i-4} as otherwise we obtain a small induced cycle with the vertices $x_2, x_3, v_{i-4}, v_{i-3}, v_{i-2}$. However $a, v_1, \ldots, v_{i-4}, x_2, x_1, v_i, \ldots, v_p, b$ is shorter than $P_{a,b}$, a contradiction. Thus the Corollary holds. $\diamond$

Lemma 3.18 *Let $S_{a,b,c}$ be a minimum ripe AT. Let $S_{x,y,z}$ be a minimum AT, with a path $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ and a center vertex u' (central vertices u', w' if of type 2) such that $S_{x,y,z} \cap (N[v_i] \setminus N(c)) \neq \emptyset$ for some $7 \leq i \leq p-6$. Then one of the following happens :*

1. *$u' \in D(a,b,c)$ ($u', w' \in D(a,b,c)$) and $P_{x,y} \cap B[a,b] \neq \emptyset$ and $P_{x,y} \cap E[a,b] \neq \emptyset$ and every v_j, $2 \leq j \leq p-1$ has a neighbor on $P_{x,y}$.*
2. *$u' \in D(a,b,c)$, ($u', w' \in D(a,b,c)$) $z \in G[a,b,c]$ and $P_{x,y} \cap G[a,b,c] = \emptyset$ and for every vertex $z' \in G[a,b,c]$, $S_{x,y,z'}$ is an AT with the same path $x, w_1, w_2, \ldots, w_q, y$.*
3. *$x \in N[v_i]$, $w_1 \in D(a,b,c)$, ($u', w_1 \in D(a,b,c)$) and $V(S_{x,y,z}) \cap P_{x,y} \cap G[a,b,c] = \{x\}$ and for every $x' \in N[v_j] \setminus N(c)$; $5 \leq j \leq p-4$, $S_{x',y,z}$ is an AT with the path $P_{x',y} = x', w_1, \ldots, w_q, y$ (See Figure 6).*

4. $y \in N[v_i]$, $w_q \in D(a,b,c)$, $(w', w_q \in D(a,b,c))$ *and* $V(S_{x,y,z}) \cap G[a,b,c] = \{y\}$ *and for every* $y' \in N[v_j] \setminus N(c)$; $5 \leq j \leq p-4$, $S_{x,y',z}$ *is an AT with the path* $P_{x,y'} = x, w_1, \ldots, w_q, y'$.

Proof: First suppose u' (u', w' if $S_{x,y,z}$ is of type 2) the center vertex (central vertices) of $S_{x,y,z}$ is in $D(a,b,c)$. Since $V(S_{x,y,z}) \cap (N[v_i] \setminus N(c)) \neq \emptyset$, we have $G[a,b,c] \cap V(S_{x,y,z}) \neq \emptyset$. Thus the conditions of the Lemma 3.16 are satisfied and hence we have (1) or (2).

Therefore we may assume that $u' \notin D(a,b,c)$ when $S_{x,y,z}$ is of type 1 and $w' \notin D(a,b,c)$ when $S_{x,y,z}$ is of type 2. Recall that $x = w_0$ and $y = w_{q+1}$.

Case 1. Suppose w_j, $0 \leq j \leq q+1$ *is in* $N[v_i] \setminus D(a,b,c)$.

Claim 3.19 $w_j \in \{w_0, w_1, w_q, w_{q+1}\}$.

Proof: For contradiction suppose $2 \leq j \leq q-1$. Note that at least one of the w_1, w_q is not in $D(a,b,c)$, as otherwise $w_1 w_q$ is an edge by Corollary 3.9. W.l.o.g assume that $w_1 \notin D(a,b,c)$. By applying Lemma 3.4 (2,3) for $S_{x,y,z}$ we have that v_i is adjacent to u' (to w') as otherwise S_{x,z,v_i} is a smaller AT and it has the condition of the Lemma. Now we have the following implications.

(f_0) $u' \in N[v_i]$, ($w' \in N[v_i]$) and (f_1) $u' \notin N(c)$ ($w' \notin N(c)$).
Otherwise by Lemma 3.4(4) for the edges $cu', u'v_i$, $(cw', w'v_i)$ $u' \in D(a,b,c)$ ($w' \in D(a,b,c)$).

(f_2) w_1 is not in $N(c)$.
Otherwise by applying Lemma 3.5 for c, w_1, u', v_i we conclude that $w_1 \in D(a,b,c)$. The same argument is applied using w' instead of u' when $S_{x,y,z}$ is of type 2.

(f_3) $x \notin N(c)$ and $x \notin D(a,b,c)$.
Since xu' (xw') is not an edge $x \notin D(a,b,c)$. This implies that $x \notin N(c)$ as otherwise by considering path c, x, w_1, u', v_i (c, x, w_1, w', v_i) and applying Lemma 3.5 we conclude that $x \in D(a,b,c)$, a contradiction.

Now by Lemma 3.17 for u', w_1, x (w', w_1, x if of type (2)) we conclude that x is adjacent to some vertex v_r, $4 \leq r \leq p-3$ and hence $x \in G[a,b,c]$. This implies that w_q is not in $D(a,b,c)$ as otherwise $w_q x$ is an edge by Corollary 3.9. By similar argument in (f_2,f_3) we conclude that $w_q, y \notin N(c)$. Now by Lemma 3.17 for u', w_q, y (w', w_q, y if of type (2)) we conclude that y is adjacent to some vertex v_r, $3 \leq r \leq p-2$ and hence $y \in G[a,b,c]$.

It remains to observe that none $w_{j-1} \notin D(a,b,c)$ as otherwise $w_{j-1}y$ would be an edge by Corollary 3.9. Similarly $w_{j+1} \notin D(a,b,c)$. Now none of the w_r, $2 \leq r \leq q-1$ is in $D(a,b,c)$ as otherwise by Corollary 3.9, $w_r x, w_r y$ are an edges of G. By similar argument in (f_2) we conclude that $w_r \notin N(c)$. Since $u'w_r$ ($w'w_r$ if of type (2)) is an edge, Lemma 3.4 (6) implies that w_r is adjacent to some v_ℓ, $i-2 \leq \ell \leq i+2$ and hence $w_r \in G[a,b,c]$. Therefore when $S_{x,y,z}$ is of type (1) we have $V(S_{x,y,z}) \subset V(G[a,b,c])$, contradicting that $S_{a,b,c}$ is ripe.

Suppose $S_{x,y,z}$ is of type (2). We observe that since $y \in G[a,b,c]$ and $yu' \notin E(G)$, $u' \notin D(a,b,c)$ by Corollary 3.6. Because $u'w_j$ is an edge $u' \in G[a,b,c]$. These imply that $V(S_{x,y,z}) \subset V(G[a,b,c])$, contradicting that $S_{a,b,c}$ is ripe. $\diamond$

We assume $w_j \in \{w_0, w_1\}$, i.e. $x \in N[v_i] \setminus N(c)$ or $w_1 \in N[v_i] \setminus N(c)$. The other case is treated similarly.

To summarize we have the following :

(a) $x \in N[v_i] \setminus N(c)$ or $w_1 \in N[v_i] \setminus N(c)$.

(b) $u' \notin D(a,b,c)$ when $S_{x,y,z}$ is of type (1) and $w' \notin D(a,b,c)$ if $S_{x,y,z}$ is of type (2).

We proceed by proving that $x \notin D(a,b,c)$ and $w_1 \in D(a,b,c)$.

Claim 3.20 *$x \notin D(a,b,c)$.*

Proof: If $j = 0$ i.e. $w_j = x$ then clearly $x \notin D(a,b,c)$. If $j = 1$ i.e. $w_j = w_1$ then we show that x is not in $D(a,b,c)$. For contradiction suppose $x \in D(a,b,c)$. Now Lemma 3.4(6) for v_iw_1, w_1w_2 implies that w_2 is in $N[v_r]$, $i-2 \le r \le i+2$ and hence $w_2 \in G[a,b,c]$. This would imply that xw_2 is an edge by Corollary 3.9, a contradiction. Therefore $x \notin D(a,b,c)$. $\diamond$

Claim 3.21 *$w_1 \in D(a,b,c)$ and $x \in N[v_i] \setminus N(c)$.*

Proof: In what follows we may assume that $S_{x,y,z}$ is of type 1. If $S_{x,y,z}$ is of type 2 we consider w' instead of u'. For contradiction suppose $w_1 \notin D(a,b,c)$. Recall items (a),(b) in summary of our assumption.

$u' \notin N(c)$. Otherwise when $w_1 \in N[v_i] \setminus N(c)$, Lemma 3.5 for path c, u', w_1, v_i implies that $u' \in D(a,b,c)$ and when $x \in N[v_i] \setminus N(c)$, Lemma 3.5 for path c, u', w_1, x, v_i implies that $u' \in D(a,b,c)$.

$x \notin N[v_i]$. For contradiction suppose $x \in N[v_i]$. Now by applying Lemma 3.5 for path c, x, v_i we conclude that $x \notin N[v_i]$ as otherwise $x \in D(a,b,c)$. Similar application of Lemma 3.5 for path c, w_1, x, v_i implies that $w_1 \notin N(c)$ as otherwise $w_1 \in D(a,b,c)$, contradiction to our assumption.

$w_1 \notin N[v_i]$. For contradiction suppose $w_1 \in N[v_i]$. Now $x \notin N(c)$ as otherwise by applying Lemma 3.5 for path c, x, w_1, v_i we conclude that x is $D(a,b,c)$. Similar application of Lemma 3.5 for path c, w_1, v_i implies that $w_1 \notin N(c)$ as otherwise $w_1 \in D(a,b,c)$, contradiction to our assumption.

We continue by having that none of the u', w_1, x is in $N(c)$.

We observe that $z \notin N(c)$ as otherwise by Lemma 3.5, one of z, u', w_1, x is a dominating vertex, contradicting (1). By Lemma 3.17 for v_i, x, w_1, u' when $x \in N[v_i]$ or Lemma 3.4 (6) for v_i, w_1, u' when $w_1 \in N[v_i]$ we conclude that $u' \in N[v_r]$, $i-3 \le r \le i+3$. Since $u'z$ is an edge and $z \notin N(c)$, z is adjacent to some vertex $v_{r'}$, $i-5 \le r' \le i+5$. W.o.l.g assume that $r' \le i$.

Now by considering the path $z, v_{r'}, v_{r'+1}, \ldots, v_i, w_j$ ($w_j \in \{w_0, w_1\}$) Lemma 3.5 implies that one of the v_ℓ, $r' \le \ell \le i$ is a dominating vertex for $S_{x,y,z}$ as otherwise we obtain a smaller AT that satisfies the condition of the lemma (in particular w_j is the same).

Now it is clear that $i-3 \leq r' \leq i$. Otherwise we get a shorter path $P'_{a,b} = a, v_1, \dots, v_{r'}, w_j, v_i, \dots, v_p, b$ when $j \neq 0$ and we get a shorter path $P''_{a,b} = a, v_1, \dots, v_{r'}, w_1, w_0, v_i, \dots, v_p, b$ when $j = 0$ (observe that we assumed that w_1 is not a dominating vertex).

Note that z is not adjacent to v_{i-5} as otherwise by Lemma 3.4(1) for $S_{x,y,z}$, v_{i-5} is adjacent to $v_{r'}$ ($i-3 \leq r'$). By applying Lemma 3.4 (7) for $v_{r'}, w_q, y$ we conclude that $y \in N[v_{\ell'}]$, $3 \leq \ell' \leq p-2$ or w_q is adjacent to v_{i-5}, v_{i-4}. However we obtain an AT, $S_{x,v_{i-5},z}$ with the path $P_{x,v_2} = x, w_1, w_2, \dots, w_q, v_{i-5}$ and center vertex u'. Since $7 \leq i \leq p-6$, $S_{x,y,z} \subset G[a,b,c]$, a contradiction. $\diamond$

We continue by having that $w_1 \in D(a,b,c)$ (a dominating vertex) and $x \in N[v_i]$. Since w_1 is not adjacent to any of the vertices $z, w_3, \dots, w_q, y$, by Lemma 3.10 none of these vertices is in $G[a,b,c]$. It is also easy to see that $u' \notin G[a,b,c]$ ($w' \notin G[a,b,c]$ when $S_{x,y,z}$ is of type 2) as otherwise because zu' is an edge Lemma 3.10 implies that w_1 is adjacent to z.

Remark : Observe that when $S_{x,y,z}$ is of type 2 then u' must be in $D(a,b,c)$. Otherwise because $v_i x, xu', u'w_3$ are edges of G by Corollary 3.17 we conclude that w_3 is adjacent to some v_r, $4 \leq r \leq p-3$ and hence $w_1 w_3$ is an edge by Lemma 3.4(7).

Finally it is easy to see that for $x' \in N(v_j) \setminus N(c)$; $5 \leq j \leq p-4$; $S_{x',y,z}$ is an AT with the path $P_{x',y} = x', w_1, \dots, w_q, y$. This proves (3). Analogously if $w_j \in \{w_q, w_{q+1}\}$, then for every $y' \in N(v_j) \setminus N(c)$; $5 \leq j \leq p-4$, $S_{x,y',z}$ is an AT with the path $P_{x,y'} = x, w_1, \dots, w_q, y'$. This shows (4).

Case 2. $z \in N[v_i] \setminus N(c)$. No vertex w_j, $0 \leq j \leq q+1$ is in $D(a,b,c)$ as otherwise $w_j z$ is an edge by Corollary 3.9. By our assumption $u' \notin D(a,b,c)$. We note that u' is adjacent to v_i by Lemma 3.4 (1). Now by applying Lemma 3.17 for v_i, u', w_1, x and for v_i, u', w_q, y we conclude that $w_1, w_q, x, y \in G[a,b,c]$. Moreover by applying Lemma 3.4 (6) for u', w_r where $2 \leq r \leq q-1$ we conclude that w_r is adjacent to some vertex v_ℓ, $i-2 \leq \ell \leq i+2$ and hence $w_r \in G[a,b,c]$. Therefore entire $S_{x,y,z}$ is in $G[a,b,c]$. This is a contradiction to $S_{a,b,c}$ is ripe. When $S_{x,y,z}$ is of type (2) $u' \notin D(a,b,c)$ as otherwise u' is adjacent to y, a contradiction. Moreover since $z \in N[v_i] \setminus N(c)$, $u' \in G[a,b,c]$ and hence $V(S_{x,y,z}) \subset V(G[a,b,c])$. $\diamond$

4 Vertex Deletion

4.1 From Chordal to Interval

In this subsection we assume that G is chordal and it does not contain small ATs. We design an FPT algorithm that takes G as an input and k as a parameter and turns G into interval graph by deleting at most k vertices. Recall that $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$, and c is a shallow vertex for $S_{a,b,c}$ and u is one of the central vertices for $S_{a,b,c}$.

Algorithm 2 Chordal-Interval(G,k)

Input : Chordal graph G without small AT's and without induced chordless cycle C, $|C| \leq 8$.
Output : A set F of G such that $|F| \leq k$ and $G \setminus F$ is interval graph OR report NO and G (i.e., no such an F, more than k vertices need to be deleted).

1. If G is an interval graph then return $\emptyset$.
2. If $k \leq 0$ and G is not interval then report NO, and return G.
3. Let $S_{a,b,c}$ be a ripe AT in G with the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (u, w when it is of type 2).
4. Let X be a smallest set of vertices such that there is no path from v_6 to v_{p-5} in $G \backslash (X \cup N(c))$ and X contains a v_j, $7 \leq j \leq p-6$.
5. If $S_{a,b,c}$ is of type (1) then if there exists a $w \in \{a, b, c, u, v_1, v_2, v_3, v_4, v_5, v_6, v_p, v_{p-1}, v_{p-2}, v_{p-3}, v_{p-4}, v_{p-5}\}$ such that $F' = Chordal - Interval(G - w, k-1)$ and $|F'| \leq k-1$ then return $F' \cup \{w\}$.
6. If $S_{a,b,c}$ is of type (2) then if there exists a $w \in \{a, b, c, u, w, v_1, v_2, v_3, v_4, v_5, v_6, v_p, v_{p-1}, v_{p-2}, v_{p-3}, v_{p-4}, v_{p-5}\}$ such that $F' = Chordal - Interval(G - w, k-1)$ and $|F'| \leq k-1$ then return $F' \cup \{w\}$.
7. Let $S = \{w' \in N[v_j] \setminus N(c); |5 \leq j \leq p-4\}$. Set $F' = Chordal - Interval(G \setminus S, k - |S|)$. If $|F' \cup S| \leq k$ then return $F' \cup S$.
8. Set $F' = Chordal - Interval(G \setminus X, k - |X|)$. If $|F' \cup X| \leq k$ then return $F' \cup X$.

The following Lemma shows the correctness of the Algorithm Chordal-Interval(G,k).

Lemma 4.1 *Let G be a chordal graph without small ATs and let $S_{a,b,c}$ be a ripe AT in G with path $P_{a,b} = a, v_1, v_2, \ldots, v_p$, and a center vertex u. Let X be a minimum separator in $G \setminus N(c)$ that separates v_6 from v_{p-5} and it contains a v_i, $7 \leq i \leq p-6$. Then there is a minimum set of deleting vertices F such that $G \setminus F$ is an interval graph and at least one of the following holds:*

(i) If $S_{a,b,c}$ is of type (1) then F contains at least one vertex from

$$\{a, b, u, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$$

If $S_{a,b,c}$ is of type (2) then F contains at least one vertex from

$$\{a, b, u, w, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$$

(ii) F contains all vertices of $S = \{x' \in N(v_j) \setminus N(c); |5 \leq j \leq p-4\}$;

(iii) F contains all the vertices in X.

Proof: Let $S_{a,b,c}$ be a ripe AT. Any optimal solution F must contains at least a vertex from $V(S_{a,b,c})$ for that reason the most interesting part of the Algorithm that should be justified is the step (7). Let H be a minimum set of deleting vertices such that $G \backslash H$ is an interval graph. We may assume that H does not contain all the vertices in S. Otherwise we set $F = H$. Moreover we may assume that H does not contain any vertex from $\{a, b, u, c, v_1, v_2, v_3, v_4, v_5, v_6, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}, v_p\}$ and if $S_{a,b,c}$ is of type (2) we may assume that H does not contain w (the other center vertex of $S_{a,b,c}$). Otherwise we set $F = H$ and we are done.

Let $W = \{w | w \in H \cap G[a, b, c]\}$. Because $S_{a,b,c}$ is an AT in G there is no path from v_6 to v_{p-5} in $G \setminus H$. Hence, set W should contain a *minimal* v_6, v_{p-5}-separator X' that contains some vertex v_j, $7 \leq j \leq p-6$ in $G \setminus N(c)$. Since $G[a, b, c]$ is an interval graph, X' is in $N[v_j]$.

We define $F = (H \setminus X') \cup X$ and we observe that $|F| \leq |H|$. Recall that $G[a, b, c]$ is an interval graph since $S_{a,b,c}$ is a ripe AT in D.

In what follows, we prove that $I = G \setminus F$ is an interval graph. For a sake of contradiction, let us assume that I is not an interval graph. By Theorem 1.1, I contains either an induced cycle of length more than three or an AT. It is clear that by deleting vertices from G no cycle would appear, since we have assumed that G is chordal. Therefore we consider the case that I may have an AT. Because we delete vertices and at the beginning G does not have small AT, we conclude that I does not have a small AT. Therefore we may assume that I contains a big AT, $S_{x,y,z}$ with the path $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ and a center vertex u'. We may assume that $S_{x,y,z}$ is of type (1). Similar argument is applied when $S_{x,y,z}$ is of type (2).

We conclude that $S_{x,y,z}$ has a vertex in X' and no vertex in X. Now according to the Lemma 3.18 one of the following happens:

1. $u' \in D(a, b, c)$ and $P_{x,y} \cap B[a, b] \neq \emptyset$ and $P_{x,y} \cap E[a, b] \neq \emptyset$ and every v_r, $2 \leq r \leq p-1$ has a neighbor on $P_{x,y}$.

2. $u' \in D(a, b, c)$, $z \in G[a, b, c]$ and $P_{x,y} \cap G[a, b, c] = \emptyset$ and for every vertex $z' \in G[a, b, c]$, $S_{x,y,z'}$ is an AT with the same path $x, w_1, w_2, \ldots, w_q, y$.

3. $x \in N[v_i]$, $w_1 \in D(a, b, c)$ and $V(S_{x,y,z}) \cap G[a, b, c] = \{x\}$ and for every $x' \in N[v_r] \setminus N(c)$; $5 \leq r \leq p-4$, $S_{x',y,z}$ is an AT with the path $P_{x',y} = x', w_1, \ldots, w_q, y$.

4. $y \in N[v_i]$, $w_q \in D(a, b, c)$ and $V(S_{x,y,z}) \cap G[a, b, c] = \{y\}$ and for every $y' \in N[v_r] \setminus N(c)$; $5 \leq r \leq p-4$, $S_{x,y',z}$ is an AT with the path $P_{x,y'} = x, w_1, \ldots, w_q, y'$.

Suppose (2) happens. Then $P_{x,y} \cap G[a, b, c] = \emptyset$ and $u' \notin G[a, b, c]$. Therefore we may assume that $z \in X'$ and no other vertex of $S_{x,y,z}$ is in $H \backslash X'$. However by (2) for every vertex $z' \in G[a, b, c]$, $S_{x,y,z'}$ is an AT with the same path $x, w_1, w_2, \ldots, w_q, y$. Since $(P_{x,y} \cup \{u'\}) \cap X' = \emptyset$ and $S_{x,y,z'}$

is not an AT in $G \setminus H$, we conclude that H must contain all the vertices in $G[a, b, c]$. This is a contradiction because $S \subset V(G[a, b, c])$ and we assumed that H does not contain entire S.

Suppose (3) happens. Then $V(S_{x,y,z}) \cap G[a, b, c] = \{x\}$ and $w_1 \in D(a, b, c)$. Therefore $P_{x,y} \cup \{u'\} \cap X' = \emptyset$. Since $X' \cap V(S_{x,y,z}) \neq \emptyset$, we have $x \in X'$. We may assume that no other vertex of $S_{x,y,z}$ is in $H \setminus X'$. However by (3) for every vertex $x' \in N[v_j] \setminus N(c)$; $5 \leq j \leq p-4$, $S_{x',y,z}$ is an AT with the path $P_{x',y} = x', w_1, \ldots, w_q, y$. Since $(P_{x,y} \cup \{u'\}) \cap X' = \emptyset$ and $S_{x',y,z}$ is not an AT in $G \setminus H$, we conclude that H must contain all the vertices in $S = \{x' \in N(v_j) \setminus N(c); |5 \leq j \leq p-4\}$. This is a contradiction because we assumed that H does not contain entire S.

Analogously (4) does not happens. Therefore the only case left to considered is (1). If (1) happens then there exists a path from v_6 to v_{p-5}. This is a contradiction to X being a separator and hence there exists some delete vertex $w' \in X$, such that $w' \in \{x, w_1, w_2, \ldots, w_q, y\}$.

$\diamond$

4.2 When G is not Chordal, Structural Properties

In this subsection we assume that G does not contain small AT, as an induced subgraph and it does not contain an induced cycle of length less than 8 and more than 3. Let $C = v_0, v_1, .., v_{p-1}, v_0$ be a shortest induced cycle in G, $8 \leq p$. We say a vertex of G is a *dominating* vertex for C if it is adjacent to every vertex of the cycle C. Let $D(C)$ denotes the set of all dominating vertices of C.

Lemma 4.2 *Let $C = v_0, v_2, .., v_{p-1}, v_0$ be a shortest induced cycle in G. Let x be a vertex in $V(G) \setminus V(C)$. Then one of the following happens :*

(1) x is adjacent to all vertices of C,

(2) x is adjacent to at most three consecutive vertices of C,

(3) Any path from $x \notin N[C]$ to C has intersection with $D(C)$.

Proof: If x is adjacent to all the vertices in $V(C)$ then (1) holds. Thus we may assume that x is not adjacent to every vertex in C.

(2) Suppose $x \in N(C)$. If x is adjacent to exactly one vertex in C then (2) holds. Therefore we may assume there are vertices $v_i \neq v_j$ of $V(C)$ such that $v_i x, v_j x$ are edges of G and none of the vertices of C between v_i and v_j in the clockwise direction is adjacent to x. We get a shorter induced cycle, using the portion of C (in the clockwise direction) from v_i to v_j and x unless up to symmetry $v_j = v_{i+1}$ or $v_j = v_{i+2}$. If $v_j = v_{i+1}$ then (2) holds. If $v_j = v_{i+2}$ then x is also adjacent

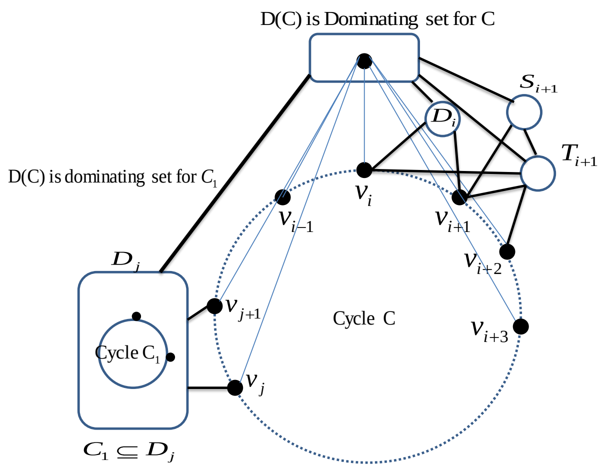

Figure 8: Cycles

to v_{i+1} as otherwise we obtain an induced 4 cycle in G which is not the case. Thus (2) is proved.

(3) For contradiction let $x \notin N(C)$ be adjacent to a vertex $y \in N(v_i) \backslash D(C)$. Now $x, y, v_{i-1}, v_i, v_{i+1}$ induce a smaller AT unless yv_{i-1}, yv_{i-2} or yv_{i+1}, yv_{i+2} are edges of G. W.l.o.g assume that yv_{i-1}, yv_{i-2} are edges of G. Now by (2) y is not adjacent to any of v_{i+1}, v_{i-3} and hence the vertices $v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_{i+1}, x, y$ induce a small AT v_{i-3}, v_{i+1}, x, a contradiction. $\diamond$

We introduce the following notations. For every $0 \leq i \leq p-1$, we define the following subsets of $N(C) \setminus D(C)$

- S_i vertices adjacent to v_i and not adjacent to any other v_j, $j \neq i$;
- D_i vertices adjacent to v_i, v_{i+1} and not adjacent to any other v_j, $j \neq i, i+1$, and
- T_i vertices adjacent to v_i, v_{i+1}, v_{i+2} only

See Figure 8 for illustration.

Lemma 4.3 *Consider the cycle C and the sets S_i, D_i, T_i, $0 \leq i \leq p-1$. Then the followings hold.*

1. *If there is an edge from a vertex in D_i to a vertex in D_j then v_i, v_j are consecutive on the cycle.*
2. *Every vertex in T_i is adjacent to every vertex in S_{i+1}.*

3. *There is no edge from S_i to $S_{i+1} \cup D_{i+1} \cup T_{i+1}$.*

Proof: (1) Let $x \in D_i$ and $y \in D_j$. Since cycle $v_{i-1}, v_i, x, y, v_{j+1}, v_{j+2}, \ldots, v_{i-2}, v_{i-1}$ is not shorter than C we have $v_j \in \{v_{i+1}, v_{i+2}, v_{i-1}, v_{i-2}\}$. We show that $v_j \neq v_{i+2}$. For contradiction suppose $v_j = v_{i+2}$. Now by definition none of the $v_{i+1}y$ and $v_{i+2}x$ is an edge of G and hence x, y, v_{i+1}, v_{i+2} induce a C_4 in G. This is a contradiction to since G does not have C_4. Similarly $v_j \neq v_{i-2}$. Therefore $v_j = v_{i-1}$ or $v_j = v_{i+1}$.

(2) Suppose $s \in S_{i+1}$ is adjacent to $t \in T_i$. Then the vertices $v_{i-1}, v_i, v_{i+1}, v_{i+2}$, v_{i+3}, s, t induce a small AT v_{i-1}, v_{i+3}, s in G. This is a contradiction because G does not have small AT as an induced subgraph.

(3) Suppose $x \in S_i$ is adjacent to $y \in S_{i+1} \cup D_{i+1} \cup T_{i+1}$. Then the vertices x, v_i, v_{i+1}, y induce a C_4, a contradiction. $\diamond$

Lemma 4.4 *Every vertex in $D(C)$ is adjacent to every vertex in $N[C]$.*

Proof: Let x be a vertex in $D(C)$ and y be a vertex in $N[v_i]$, $0 \leq i \leq p-1$. Note that if $y = v_i$ then by definition of $D(C)$, xy is an edge.

If $y \in D(C) \cup T_i$ then xy is an edge as otherwise x, y, v_i, v_{i+2} induce a C_4. Thus by Lemma 4.2 (2) we may assume that $y \in S_i \cup D_i$. This implies that y is adjacent to v_i and not adjacent to v_{i-1} and not adjacent to v_{i+2}. However v_{i-2}, v_{i+2}, y is a small AT, with the vertices $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}, x, y$. $\diamond$

4.2.1 Cycle-Cycle interaction

Lemma 4.5 *Let C_1 be an induced cycle in $N[C] \setminus V(C)$ with length at least 9. Then $V(C_1) \cap D(C) = \emptyset$ and one of the following happens:*

1. *For every $0 \leq i \leq p-1$, $N(v_i) \cap V(C_1) \neq \emptyset$.*
2. *$V(C_1) \subset S_i$ or $V(C_1) \subset V(D_i)$ (See Figure 8).*

Proof: Observe that $|V(C_1)| \geq 4$. By Lemma 4.4 every vertex x in $D(C)$ is adjacent to every vertex in $N[C] \setminus \{x\}$, we conclude that $D(C) \cap V(C_1) = \emptyset$.

Suppose (1) does not hold. Thus there exists some i such that $V(C_1) \cap N(v_i) = \emptyset$ but $V(C_1) \cap N(v_{i+1}) \neq \emptyset$.

Let $v \in V(C_1) \cap N(v_{i+1})$. Let x be the last vertex of C_1 after v in the clockwise direction such that $x \in N(v_{i+1})$ but x' the neighbor of x in C_1 (clockwise direction) is not in $N(v_{i+1})$. We

show that if x exists then we obtain a small AT. Let y' be the first vertex after x in the clockwise direction such that $y' \in N(v_{i+1})$ but y the neighbor of y' in C_1 (clockwise direction) is in $N(v_{i+1})$.

Note that if x exists then y also exists. We note that none of the x', y' is adjacent to v_{i-1} as otherwise we obtain an induced small cycle with the vertices $v_{i-1}, v_i, v_{i+1}, x, x'$ or with the vertices $v_{i-1}, v_i, v_{i+1}, y, y'$.

Observe that $x' \neq y'$ as otherwise v_{i+1}, x, y, x' induce a C_4. Moreover $x'y'$ is not an edge of G as otherwise the vertices y', x', x, v_{i+1}, y induce a C_5. However we get a small AT v_{i-1}, x', y' with the vertices $x, y, v_{i+1}, v_i, v_{i-1}, x', y'$. Therefore x does not exist and hence $V(C_1) \subseteq D_{i+1} \cup S_{i+1} \cup T_{i+1}$.

First suppose $V(C_1) \cap S_{i+1} \neq \emptyset$ and $V(C_1) \cap (D_{i+1} \cup T_{i+1}) \neq \emptyset$. If $|V(C_1) \cap S_{i+1}| = 1$ then we get an induced cycle x, x', y', v_{i+2} where $x \in V(C_1) \cap S_{i+1}$ and x', y' are the neighbors of x in $V(C_1) \cap D_{i+1} \cup T_{i+1}$.

Similarly if $|V(C_1) \cap D_{i+1} \cup T_{i+1}| = 1$ we get an induced C_4 in G. Thus we may assume that C_1 has at least two vertices in S_{i+1} and two vertices in $D_{i+1} \cup T_{i+1}$. Let $x \neq y$ be two vertices of C_1 in $D_{i+1} \cup T_{i+1}$. Now let xx', yy' be the edges of C_1 such that $x', y' \in S_{i+1} \setminus D_{i+1} \cup T_{i+1}$. If $x' = y'$ we obtain an induced 4 cycle with the vertices x', x, y, v_{i+2} (xy is not an edge as otherwise $|C_1| \leq 4$.) in G. If $x'y'$ is an edge then we obtain induced 5 cycle in G with the vertices x', y', x, y, v_{i+2} in G. Now we obtain a small AT v_{i+4}, x', y' with the vertices $x', y', x, y, v_{i+2}, v_{i+3}, v_{i+4}$. This is a contradiction and hence we have $V(C_1) \subset S_{i+1}$ or $V(C_1) \subseteq D_{i+1} \cup T_{i+1}$. If $V(C_1) \subset S_{i+1}$ then (2) is proved. Thus we may assume that C_1 has vertices in T_{i+1} and D_{i+1} only. Note that $|V(C_1) \cap T_{i+1}| \leq 2$. Again by similar argument and considering v_{i+3}, v_{i+4} and part of C_1 inside D_{i+1} we see a small AT. Therefore $V(C_1) \subset V(D_{i+1})$ and the proof is complete. $\diamond$

Definition 4.6 *We say a shortest cycle $C = v_0, v_1, \ldots, v_{p-1}, v_0$ in G is* clean *if there is no $0 \leq i \leq p-1$ such that $N(v_i) \setminus V(C)$ contains an induced cycle of length more than 3. We say a cycle C is* ripe *if it is clean and there is no AT, $S_{x,y,z}$ in $N[C] \setminus D(C)$.*

Looking for a shortest clean cycle

We start with an arbitrary shortest cycle C and we construct $S_i, D_i, T_i, D(C)$ as defined and then we look for a shortest cycle C_1 in some S_i or D_i. If C_1 is clean we stop otherwise we consider S_i^1, D_i^1, T_i^1 of C_1 in D_i or S_i and we continue. After at most k steps we find a clean cycle C'.

4.2.2 Cycle and AT interaction

Lemma 4.7 *Let $S_{x,y,z}$ be a minimum AT with a path $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ such that $S_{x,y,z}$ contains a vertex from $D(C)$ and a vertex from $N[C] \setminus D(C)$.*

Then u (center vertex of $S_{x,y,z}$) is a dominating vertex for C, and none of the vertices

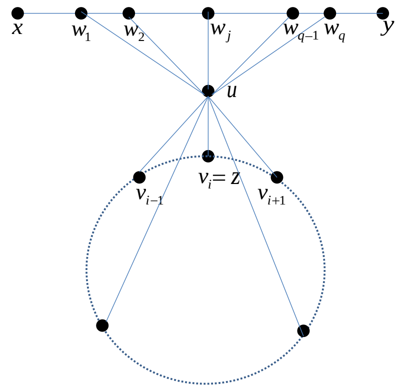

Figure 9: Cycle and AT outside

$x, w_1, \dots, w_q, y$ is in $N[C]$, and $z \in N[C] \setminus D(C)$. Moreover for every vertex $z' \in N[C] \setminus D(C)$, $S_{x,y,z'}$ is an AT with the same number of vertices as $S_{x,y,z}$ and the same path $P_{x,y}$ (Figure 9).

Proof: First suppose $P_{x,y} \cap (N[C] \setminus D(C)) \neq \emptyset$.

We show that $x \in N[C]$. For contradiction suppose $x \notin N[C]$. Let $1 \leq i \leq q+1$ be the first index such that $w_i \in N[C]$. By Lemma 4.2 (3) it is easy to see that w_i is a dominating vertex for C and hence none of the vertices $w_{i+2}, w_{i+3}, \dots, w_q, y$ is in $N[C]$ by Lemma 4.4. Note that by assumption for i, none of the $x, w_1, \dots, w_{i-2}$ is in $N[C]$. Since u is adjacent to $z, w_1, w_2, \dots, w_q$, we conclude that u must be in $D(C)$. This implies that $z \in N[C] \setminus D(C)$. Now by Lemma 4.4 zw_i is an edge, a contradiction. Therefore $x \in N[C]$. Analogously $y \in N[C]$. Now we conclude that $u, z \notin D(C)$ as otherwise by Lemma 4.4, xu (uy if $S_{x,y,z}$ is of type (2)) or zx is an edge. Thus some vertex w_i is in $D(C)$ and hence xw_i and yw_i is an edge, yielding a contradiction. Therefore we conclude that $P_{x,y} \cap N[C] \setminus D(C) = \emptyset$.

Now it is easy to see that since u is adjacent to $w_1, w_2, \dots, w_q$ we have $u \in D(C)$ and hence for every $z' \in (N[C] \setminus D(C))$, $S_{x,y,z'}$ is a minimum AT with the same number of vertices as $S_{x,y,z}$. $\diamond$

Lemma 4.8 *Let $S_{x,y,z}$ be a minimum AT such that the path $P_{x,y} = x, w_1, w_2, \dots, w_q, y$ is in $N[C] \setminus D(C)$. Then there exists a $v_i \in V(C)$, $0 \leq i \leq p-1$ such that v_i is a dominating vertex for $S_{x,y,z}$. Moreover none of the v_{i-2}, v_{i+2} is adjacent to any w_j, $2 \leq j \leq q-1$ (See Figure 10).*

Proof: Observe that by Lemma 4.4 none of the u, z (u, w, z if $S_{x,y,z}$ is of type (2)) is in $D(C)$ as otherwise one of the xu, zx (wx, zx if of type (2)) is an edge. Therefore $V(S_{x,y,z}) \subset N[C] \setminus D(C)$.

First suppose $u = v_i$ for some $0 \leq i \leq p-1$. Now v_{i-2} is not adjacent to w_j, $2 \leq j \leq q-1$ as otherwise by Lemma 3.4(2,3) uv_{i-2} is an edge, a contradiction. Similarly v_{i+2} is not adjacent to w_j. In this case the proof is complete. Therefore we continue by assuming that $u \notin V(C)$.

We show that $z \notin V(C)$. For contradiction suppose $z = v_i$, $0 \leq i \leq p-1$. Since $u \notin V(C)$, we assume that $u \in S_i \cup D_i \cup T_i$. By Lemma 3.4 (1) both $v_{i-1}u$ and $v_{i+1}u$ are edges of G. Now v_{i-1}

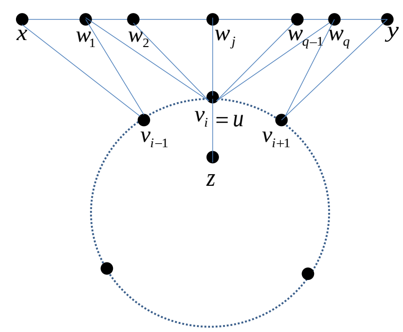

Figure 10: Cycle and AT inside N[C]

is adjacent to some vertex w_j, $0 \leq j \leq q+1$. Otherwise $S_{a,b,v_{i-1}}$ is a minimum AT, with the same number of vertices as $S_{a,b,c}$ and hence by Lemma 3.4 (1) v_{i-2} must be adjacent to u implying that u is adjacent to more than three consecutive vertices on the cycle. This is a contradiction to Lemma 4.2(2). Therefore v_{i-1} is adjacent to some w_j w_j, $0 \leq j \leq q+1$ and hence by Lemma 3.4 (4), v_{i-1} is a dominating vertex for $S_{x,y,z}$. Similarly we conclude that v_{i+1} is a dominating vertex for $S_{x,y,z}$ and hence by Corollary 3.6 $v_{i-1}v_{i+1}$ must be an edge. This is a contradiction. Therefore we conclude the following :

(f_1) For every minimum AT, $S_{x',y',z'}$ such that $V(S_{x',y',z'}) \subset N[C] \setminus D(C)$ we have $z' \notin V(C)$.

We continue by assuming that $z \in N(v_i) \setminus V(C)$, $0 \leq i \leq p-1$. Now $v_i u$ is also an edge by Lemma 3.4(1). Note that v_i is adjacent to some vertex w_j, $0 \leq j \leq q+1$ on the path $P_{x,y}$ as otherwise S_{x,y,v_i} is a minimum AT with the same number of vertices as $S_{x,y,z}$ and we get a contradiction by (f_1). Since zv_i is an edge and v_i is adjacent to w_j, Lemma 3.4(4) implies that v_i is a dominating vertex for $S_{x,y,z}$. Now v_{i-2} is not adjacent to any w_r, $2 \leq r \leq q-1$ as otherwise by Lemma 3.4(7) v_{i-2} must be adjacent to v_i which is not possible. Similarly v_{i+2} is not adjacent to any w_r, $2 \leq r \leq q-1$. $\diamond$

4.3 The Main Algorithm (Putting things together)

We branch on all the deleting vertices of each small AT. We also branch on by deleting vertices of each induced cycle C, $4 \le |C| \le 8$. After that if G is not interval we continue as follows.

Definition 4.9 *Let C be a ripe cycle. We say a set X of the vertices in $N[C] \setminus D(C)$ is a cycle-separator if there is no cycle in $N[C] \setminus (D(C) \cup X)$.*

Observe that when C is ripe, $G[N(C) \cup V(C) \setminus D(C)]$ is a *circular arc graph* [22] and finding a cycle-separator in a circular arc graph is equivalent to finding a minimal clique which is polynomial time solvable.

Algorithm 3 Interval-Deletion(G,k)

Input : Graph G without small AT's and without induced chordless cycle C, $4 \le |C| \le 8$.
Output : A set F of $V(G)$ such that $|F| \le k$ and $G \setminus F$ is interval graph OR report No and G (i.e., no such F, more than k vertices need to be deleted).

1. If G is an interval graph then return $\emptyset$.
2. If $k \le 0$ and G is not an interval graph then report NO and return G.
3. If G is chordal then set $F = Chordal - Interval(G, k)$. If $|F| \le k$ then return F otherwise report NO and return G.
4. Let C be a shortest ripe cycle in G. Let X be a minimum cycle-separator in $N[C] \setminus D(C)$.
5. Set $F = Interval - Deletion(G \setminus C, k - |C|)$. If $|F \cup C| \le k$ then return $F \cup C$. Else report NO and return G.
6. Set $F = Interval - Deletion(G \setminus X, k - |X|)$. If $|F \cup X| \le k$ then return $F \cup X$. Else report NO and return G.
7. Let $C = v_0, v_1, \ldots, v_{p-1}$ be a shortest clean cycle in G.
8. Let $S = \{x | x \in (N[v_{i-1}] \cup N[v_i] \cup N[v_{i+1}]) \setminus D(C)\}$ for some $0 \le i \le p - 1$ such that $G[S]$, contains an AT.
9. Set $F = Chordal - Interval(G[S], k)$. Set $F' = Interval - Deletion(G \setminus F, k - |F|)$. If $|F' \cup F| \le k$ then return $F \cup F'$. Else report NO and return G.

If there is no cycle in G then we apply the Chordal-to-Interval Algorithm and as we argued in Lemma 4.1 there is an optimal solution that contains the solution of the Chordal-to-Interval Algorithm. Otherwise let C be a clean cycle in G. If C is ripe then we argue in Lemma 4.10 that there exists a minimum set X of the vertices in $N[C] \setminus D(C)$ and there is minimum set of deleting vertices F such that $G \setminus F$ is an interval graphs and $X \subseteq F$. Therefore the steps 4,5 are justified. If C is not ripe then there is some AT', $S_{x,y,z}$ such that $V(S_{x,y,z}) \cap (N[C] \setminus D(C)) \neq \emptyset$, and $V(S_{x,y,z}) \subseteq W = N[v_{i-1}] \cup N[v_i] \cup N[v_{i+1}]$ for three consecutive vertices v_{i-1}, v_i, v_{i+1} in C.

We apply the Algorithm Chordal-to-Interval on the subgraph induced by W and hence we ripen the cycle C. The correctness of step 7,8 are justified by Lemma 4.11.

Overall the running time of the algorithm is $O(c^k n(m+n)$ where $c = min\{18, k\}$. In order to find an AT we apply the algorithm in [14]. Now we focus on the correctness of the Interval-Deletion(G,k) algorithm.

Lemma 4.10 *Let $C = v_0, v_1, \dots, v_{p-1}, v_0$ be a ripe cycle and Let X be a minimum cycle-separator in $N[C] \setminus D(C)$. Then there is a minimum set of deleting vertices F such that $G \setminus F$ is an interval graph and at least one of the following holds:*

(i) F contains all the vertices of the cycle C.

(ii) F contains all vertices in X.

Proof: Let H be a minimum set of deleting vertices such that $G \setminus H$ is an interval graph. If H contains all the vertices in C then we set $F = H$ and we are done. Thus we suppose H does not contain all the vertices of C. Let $W = \{w | w \in H \cap (N[C] \setminus D(C))\}$.

Since H does not contain all the vertices of C, set W should contain a *minimal* cycle-separator X' in $N[C] \setminus D(C)$. Now define $F = (H \setminus X') \cup X$. We observe that $|F| \leq |H|$. We prove that $I = G \setminus F$ is an interval graph.

First suppose I contains an induced cycle C_1. We note that $V(C_1) \cap X' \neq \emptyset$ and $V(C_1) \cap X = \emptyset$. Since C is ripe, by Lemma 4.5 (1) for every $0 \leq i \leq p-1$, $N[v_i] \cap V(C_1) \neq \emptyset$. But this a contradiction to X being a cycle-separator.

Therefore we may assume that I contains an AT. Since $G \setminus F$ has less vertices than G and G does not have small ATs, I does not have small ATs. Thus we may assume that I contains a big AT. Consider a big AT, $S_{x,y,z}$ with the $P_{x,y} = x, z_1, z_2 \dots, z_m, y$ in I. We note that $S_{x,y,z}$ must have some vertices in $X' \setminus X$ and none of the vertices of $S_{x,y,z}$ is in $F \setminus X'$. Since $X' \subset N[C] \setminus D(C)$ and cycle C is ripe the conditions of the Lemma 4.7 occur. According to Lemma 4.7, $P_{x,y} \cap N[C] = \emptyset$ and vertex u (one of the central vertices of $S_{x,y,z}$) is in $D(C)$ and $z \in N[C] \setminus D(C)$. This implies that $z \in X'$. Moreover by Lemma 4.7 for every vertex $z' \in N[C] \setminus D(C)$, $S_{x,y,z'}$ is a minimum AT with the same number of vertices as $S_{x,y,z}$, and in particular for every vertex v_i, $0 \leq i \leq p-1$ of C, S_{x,y,v_i} is a minimum AT with the same number of vertices as $S_{x,y,z}$. Since S_{x,y,v_i} is no longer an AT in $G \setminus H$, $u \notin H$, and $V(S_{x,y,v_i} \setminus \{v_i\}) \cap (N[C] \setminus D(C)) = \emptyset$, we conclude that H must contain v_i. Therefore H must contain all the vertices in $V(C)$, a contradiction.

$\diamond$

Lemma 4.11 *Let $C = v_1, v_2, \dots, v_p$ be a clean cycle. Let $S_{a,b,c}$ be a ripe AT in $N[C] \setminus D(C)$, with the path $P_{a,b} = a, w_1, w_2, \dots, w_q, b$. Let X be a minimum separator in $G \setminus N(c)$ that separates w_6 from w_{q-5} and it contains a vertex w_i, $7 \leq i \leq q-6$. Then there is a minimum set of deleting vertices F such that $G \setminus F$ is an interval graph and at least one of the following holds:*

(i) F contains at least one vertex from

$$\{a, b, u, c, w_1, w_2, w_3, w_4, w_5, w_6, w_{q-5}, w_{q-4}, w_{q-3}, w_{q-2}, w_{q-1}, w_q\}$$

(ii) F contains all vertices of $S = \{w' \in N[w_j] \setminus N(c); |5 \leq j \leq p-4\}$;

(iii) F contains all the vertices in X.

Proof: The proof is similar to the proof of Lemma 4.1. Let H be a minimum set of deleting vertices such that $G \setminus H$ is an interval graph. The only extra argument needed is the following. The only way that the selection of X may affect the optimal solution H is when some of the vertices in a minimal separator X' (that separate w_6 from w_{q-5} outside the neighborhood of c) are used in order to break a shortest cycle in $N[C]$. Since C is a clean cycle, every cycle C' in $N[C]$ contains a vertex from $N[v_r]$, $0 \leq r \leq p-1$. We define S'_i be the set of vertices in $G \setminus N(c)$ that are adjacent to w_j only. Similarly we define D'_j and T'_j. According to Lemma 4.8, there exists some $v_i \in V(C)$ such that v_i is a dominating vertex for $S_{a,b,c}$. This allows us to be able to apply the Lemma 3 for a subgraphs of G that does not have any cycle and is contained in the vertices of v_{i-1}, v_i, v_{i+1} together with their neighborhood. We conclude that $G[a,b,c]$ is in neighborhood of v_i. It is easy to see that C' does not contain any vertex from $D(a,b,c)$ as otherwise we get a shorter cycle than C'. Since C' contains a vertex from $N[v_{i-2}]$ and a vertex from $N[v_{i+2}]$, C' must enter $G[a,b,c]$ from $B[a,b]$ and leaves $G[a,b,c]$ from $E[a,b]$. Therefore C' has a vertex from X. $\diamond$

5 Interval Completion

5.1 AT and AT Edge Interaction

In this subsection we may assume that G is chordal and G does not contain any small AT.

Lemma 5.1 *Let $S_{a,b,c}$ be a minimum AT with path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (central vertices u, w). Let $Q = u, x_1, x_2, \ldots, x_r, a$ be a chordless path from u to a (not including v_1) (from w to a if $S_{a,b,c}$ is of type 2). Then for every vertex x_i, $2 \leq i \leq r$, $S_{x_i,b,c}$ is a minimum AT with the same number of vertices as $S_{a,b,c}$, and path $P_{x_i,b} = x_i, v_1, v_2, \ldots, v_p, b$*

Proof: Since there is no induced cycle of length more than 3 in G, v_1 must be adjacent to x_i, $1 \leq i \leq r$. Now c is not adjacent to any x_i, $2 \leq i \leq r$. Otherwise by item (4) of Lemma 3.4 (cx_i, x_iv_1 are edges) x_i is a dominating vertex for $S_{a,b,c}$ and hence by Corollary 3.6 x_i would be adjacent to u (w) contradiction to Q being chordless. We note that x_i is not adjacent to v_j, $2 \leq j \leq p+1$ as otherwise we get a smaller AT $S_{x_i,b,c}$ with the path $x_i, v_j, v_{j+1}, \ldots, v_p, b$. If $S_{x,y,z}$ is of type (2) we note that u is adjacent to x_i as otherwise we get an AT with the vertices $c, u, x_i, v_1, v_2, \ldots, v_p, b$ and has fewer vertices than $S_{a,b,c}$. Thus cu is an edge when $S_{a,b,c}$ is of type (2). Now regardless of type of $S_{a,b,c}$ we conclude that $S_{x_i,b,c}$ is also a minimum AT with path $P_{x_i,b} = x_i, v_1, \ldots, v_p, b$. $\diamond$

Analogous to the Lemma 5.1 we have the following Lemma.

Lemma 5.2 *Let $S_{a,b,c}$ be a minimum AT with path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (central vertices u, w). Let $Q = u, x_1, x_2, \ldots, x_r, b$ be a chordless path from u to b (not including v_p) (from u to b if $S_{a,b,c}$ is of type 2). Then for every vertex x_i, $2 \leq i \leq r$, $S_{a,x_i,c}$ is a minimum AT with the same number of vertices as $S_{a,b,c}$, and path $P_{a,x_i} = a, v_1, v_2, \ldots, v_p, x_i$.*

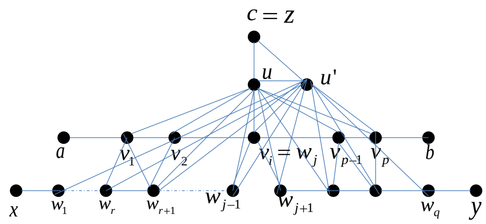

Figure 11: $u' \in D(a,b,c)$ and $P_{x,y} \cap B[a,b] \neq \emptyset$, $P_{x,y} \cap E[a,b] \neq \emptyset$

Definition 5.3 *Let $S_{a,b,c}$ be an AT in graph G. We refer to a fill-in edge cv_i, $0 \le i \le p+1$, as a* long fill-in edge *and we refer to a fill-in edge v_iv_j, $0 \le i,j \le p+1$, as a* bottom fill-in edge *of $S_{a,b,c}$. Note that ac, bc are long fill-in edges when $S_{a,b,c}$ is of type 2 and that ab is a bottom fill-in edge.*

By cross fill-in edges *of $S_{a,b,c}$, we call fill-in edges au, bu when $S_{a,b,c}$ is of type 1 and aw, bu, when $S_{a,b,c}$ is of type 2.*

Let us remark that in a graph G' obtained from G by adding either long or cross fill-in edge, subgraph $S_{a,b,c}$ does not induce a cycle of length more than 3 and it does not induce an AT.

Lemma 5.4 *Let $S_{a,b,c}$ be a minimum ripe AT. Let $S_{x,y,z}$ be a minimum AT, with path $P_{x,y} = x, w_1, \ldots, w_q, y$ such that a long fill-in edge cd, $d \in G[a,b,c]$ is a fill-in edge of $S_{x,y,z}$. Then cd is a long fill-in edge of $S_{x,y,z}$ and one of the following happens :*

1. *$z = c$, $P_{x,y} \cap B[a,b] \neq \emptyset$, $P_{x,y} \cap E[a,b] \neq \emptyset$, and every v_i, $2 \le i \le p-1$ has a neighbor on $P_{x,y}$ (See Figure 11).*
2. *$z = d$, and for every vertex $z' \in G[a,b,c]$, $S_{x,y,z'}$ is an AT with the same path $P_{x,y} = x, w_1, w_2, \ldots, w_q, y$ (See Figure 12).*

Proof: By definition of $G[a,b,c]$, d is adjacent to some v_i, $3 \le i \le p-2$. We first show that ab is not a bottom fill-in edge for $S_{x,y,z}$. Otherwise up to symmetry we may assume that $x = c$ and $y = d$. Now by Lemma 3.12 for the path $c, w_1, w_2, \ldots, w_q, y$ we conclude that w_1 is a dominating vertex for $S_{a,b,c}$ and hence by Lemma 3.4 (7) w_1y is an edge of G, yielding a contradiction.

We show that cd is not a cross fill-in edge for $S_{x,y,z}$. For contradiction suppose cd is a cross fill-in edge for $S_{x,y,z}$. Let u' be one of the center vertices of $S_{a,b,c}$. Now consider the path c, x', d that corresponds to one the paths x, w_1, u' and y, w_q, u' and u', w_1, x and u', w_1, y in $S_{x,y,z}$. By Lemma 3.12 for the path c, x', d we conclude that x' is a dominating vertex for $S_{a,b,c}$. W.l.o.g

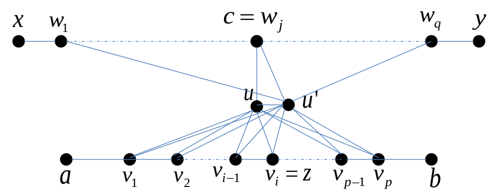

Figure 12: $z \in G[a,b,c]$ and $P_{x,y} \cap G[a,b,c] = \emptyset$

assume that c, x', d corresponds to path u', w_1, x or path x, w_1, u'. Thus w_1 is a dominating vertex for $S_{a,b,c}$. We observe that $u' \neq c$ as otherwise because $u'z = cz$ is an edge, Corollary 3.6 implies that zw_1 is an edge. Therefore we have $c = x$ and $u' = d$. Because $u' \in G[a,b,c]$ and $u'w_3$ is an edge by Lemma 3.10 w_3 is adjacent to every vertex in $D(a,b,c)$ and in particular w_3w_1 is an edge; a contradiction. Therefore cd is not a cross fill-in edge for $S_{x,y,z}$.

We conclude that cd is a long fill-in edge. By considering the path c, u', d and applying the Lemma 3.12 we conclude that u' is a dominating vertex for $S_{a,b,c}$. Now the conditions of the Lemma 3.16 are satisfied and hence one of the (1),(2) holds.

$\diamond$

Lemma 5.5 *Let $S_{a,b,c}$ be a minimum ripe AT with path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let $S_{x,y,z}$ be a minimum AT, with center vertex u' (central vertices u', w' if of type 2). If bottom fill-in edge ab is a fill-in edge of $S_{x,y,z}$ with path $P_{x,y} = x, w_1, \ldots, w_q, y$ then one of the following happens :*

1. *ab is a cross fill-in edge, $a = x$, $b = u'$ and for every v_i, $1 \leq i \leq p$, $S_{v_i,y,z}$ is an AT with the same number of vertices as $S_{x,y,z}$ and the same center vertex u'.*

2. *ab is a cross fill-in edge, $a = y$, $a = u'$ and for every v_i, $1 \leq i \leq p$, $S_{x,v_i,z}$ is an AT with the same number of vertices as $S_{x,y,z}$ and the same center vertex u'.*

3. *$S_{x,y,z}$ is of type (2) and $a = x$, $b = z$ and for every v_i, $1 \leq i \leq p$, $S_{v_i,y,z}$ is an AT with the same number of vertices as $S_{x,y,z}$ and the same central vertices u', w'.*

4. *$S_{x,y,z}$ is of type (2) and $a = z$, $b = y$ and for every $y' \in G[a,b,c]$, $S_{x,v_i,z}$ is an AT with the same number of vertices as $S_{x,y,z}$ and the same central vertices u', w'.*

Proof: First suppose ab is a cross fill-in edge of $S_{x,y,z}$. Therefore up to symmetry we are left with the case $a = x$ and $b = u'$. By Lemma 5.1 for $S_{x,y,z}$ we conclude that for every v_i, $S_{v_i,y,z}$ is an AT with the same number of vertices as $S_{x,y,z}$.

Now suppose that ab is a long fill-in edge for $S_{x,y,z}$. If $a = w_j$ for some $1 \leq j \leq q$ and $b = z$ then by Lemma 3.5 for $S_{x,y,z}$ and path $z, v_p, v_{p-1}, \ldots, v_1, a$ we conclude that v_p is a dominating vertex for $S_{x,y,z}$ and hence $v_p v_1$ is an edge. This is a contradiction. Thus up to symmetry we may assume that $a = x$ and $b = z$. We note that in this case $S_{x,y,z}$ is of type (2). Now again by Lemma 3.5, v_p is a dominating vertex for $S_{x,y,z}$. We note that v_p is not adjacent to a and hence v_p must be adjacent to y otherwise we obtain a smaller AT with the vertices $z, v_p, x, w_1, \ldots, w_q, y$, contradicting the minimality of $S_{x,y,z}$. Observe that by replacing w' with v_p we obtain an AT $(S_{x,y,z})'$ with the same number of vertices as $S_{x,y,z}$. However by applying Lemma 5.2 for $(S_{x,y,z})'$ with the path $v_p, v_{p-1}, ...v_1, x$ we conclude that for every v_i, $S_{v_i,y,z}$ is an AT with the same number of vertices as $S_{x,y,z}$.

$\diamond$

5.2 Algorithm For Interval Completion

Interval-Completion(G,K)
Input : Graph G, and parameter k.
Output : A set F of edges such that $|F| \leq k$ and $G \cup F$ is interval graph OR report NO and return M (M is a fixed set of more than k edges).

1. If G is an interval graph then return $\emptyset$.

2. If $k \leq 0$ and G is not interval graph then report NO solution, and return M

3. Let C be an induced cycle with $|C| \geq 4$. For every minimal triangulation F of C set $F' = Interval - Completion(G \cup F, k - |C| + 3)$. If $|F \cup F'| \leq k$ then return $F \cup F'$.

4. Let S be a small AT in G. For every edge e (at most 9 ways) such that $S \cup \{e\}$ is not an AT in G set $F = Interval - Completion(G \cup \{e\}, k - 1)$. If $|F| \leq k - 1$ then return $F' \cup \{e\}$.

5. Let S_{a_0,b_0,c_0} be a minimum AT in G. Apply the Algorithm 1 to obtain a minimum ripe AT, $S_{a,b,c}$ with the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center vertex u (u, w when is of type 2).

6. Let X be a smallest set of vertices such that X contains a vertex $v_i \in X$, $7 \leq i \leq p - 6$, and there is no path from v_6 to v_{p-5} in $G \setminus (X \cup N(c))$.

7. Let $S = \{au, bu\}$ when $S_{a,b,c}$ is of type 1 otherwise let $S = \{aw, bu\}$. Set $F = Interval - Completion(G \cup \{e\}, k - 1)$. If $|F| \leq k - 1$ then return $F \cup \{e\}$.

8. If $S_{a,b,c}$ is of type 1 then for **each** of the long fill edge $e = cv_i$, $i \in \{1, 2, 3, 4, 5, 6, p - 5, p - 4, p - 3, p - 2, p - 1, p\}$ set $F = Interval - Completion(G \cup \{e\}, k - 1)$. If $|F| \leq k - 1$ then return $F \cup \{e\}$.

9. If $S_{a,b,c}$ is of type 2 then for **each** of the long fill edge $e \in ca, cv_i, cb$, $i \in \{1, 2, 3, 4, 5, 6, p - 5, p - 4, p - 3, p - 2, p - 1, p\}$ set $F = Interval - Completion(G \cup \{e\}, k - 1)$. If $|F| \leq k - 1$ then return $F \cup \{e\}$.

10. Let $S_1 = \{av_i | 2 \leq i \leq p\} \cup \{ab\}$. Set $F = Interval - Completion(G \cup S_1, k - p)$. If $|F \cup p| \leq k$ then return $F \cup S_1$.

11. Let $S_2 = \{bv_i | 1 \leq i \leq p-1\} \cup \{ab\}$. Set $F = Interval - Completion(G \cup S_2, k-p)$. If $|F \cup p| \leq k$ then return $F \cup S_2$.

12. Let U be the set of vertices adjacent to u and not adjacent to any vertex on the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let C be a connected component of $G[U]$, containing c

13. Let S_3 be the set of all edge $c'x$ for some $c' \in C$ and $x \in X$. Set $F = Interval - Completion(G \cup S_3, k - |X|)$. If $|F \cup S_3| \leq k$ then return $F \cup S_3$.

We now focus on the correctness of the Interval-Completion algorithm. In what follows we consider the minimum ripe AT, $S_{a,b,c}$ with the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$ and center (central vertices) u (u, w if of type 2).

Definition 5.6 *For center vertex u in $S_{a,b,c}$ let U be the set of vertices adjacent to u and not adjacent to any vertex on the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let C be a connected component of $G[U]$, containing c (by Lemma 3.4 (4) for every $c' \in C$, $S_{a,b,c'}$ is a minimum AT, with the same number of vertices as $S_{a,b,c}$).*

Recall that $G[a,b,c] = \{x | x \in xv_i \in E(G), 3 \leq i \leq p-2\}$. According to the definition of ripe AT, $G[a,b,c]$ is an interval graph. Now we are ready to prove the following Lemma.

Lemma 5.7 *Let X' be a minimal separator in $G[a,b,c]$ such that $v_i \in X'$, $7 \leq i \leq p-6$. Set $E'_X = \{c'x' | c' \in C, x' \in X'\}$. Then $G \cup E'_X$ does not contain a minimum AT, S (small or big) containing some edges of E'_X.*

Proof: For contradiction we assume there exists a minimum AT, that uses one edge from E'_X. We get a contradiction by showing that there exists another AT in G and it has a vertex from $N[v_i]$, $7 \leq i \leq p-6$ but none of the items (1,2,3,4) of the Lemma 3.18 holds for this AT and consequently $G[a,b,c]$ is not an interval graph i.e. $S_{a,b,c}$ is not ripe.

Let S be a minimum AT with the vertices x', y', z' such that cd, $c \in C$ and $d \in X'$ is an edge of S. We assume that $d \in N[v_i]$, $7 \leq i \leq p-6$. W.l.o.g assume that cd is an edge of the shortest path P_1 from x' to y' outside the neighborhood of z'. We observe that $z' \notin C$ as otherwise $z'd$ is an edge. Since d is adjacent to every vertex $c' \in C$, we conclude that P_1 has only one vertex c in C.

Let d_1 be the neighbor of d in P_1 and d_2 be the neighbor of c in P_1. First suppose $c \neq y'$. Now d_2 is not in $D(a,b,c)$ as otherwise by Corollary 3.6 dd_2 is an edge , a contradiction to the minimality of the length of P_1. $d_2 \notin C$ as otherwise dd_2 is an edge in $E_{X'}$ and we get a shorter path. Thus we conclude that $d_2 \in X'$ and since X' is a clique, dd_2 is an edge and hence we replace dc, cd_2 by dd_2 in P_1 and we get a shorter path. Therefore we assume that $c = y'$.

We note that $P_1 \cap D(a,b,c) = \emptyset$ as otherwise we get a shorter path from x' to y'. Let P_2 be the shortest path from y' to z' that avoids the neighborhood of x' and P_3 be the shortest path from z' to x' that avoids the neighborhood of y'. By Corollary 3.6 $P_3 \cap D(a,b,c) = \emptyset$. Because $y' = c$ by Corollary 3.6 none of the $x', z' \in D(a,b,c)$. We also note that P_3 does not uses any edge $z_1 z_2$, $z_2 \in X'$ as otherwise by $y'z_2$ is an edge in $E_{X'}$.

We show that P_2 goes through some vertex in $D(a,b,c)$. For contradiction suppose $P_2 \cap D(a,b,c) = \emptyset$. If P_2 does not use any of the edges in E'_X then P_2P_3 contains an induce shortest path Q from $c = y'$ to vertex $d \in G[a,b,c]$. Thus lemma 3.12 implies that d_2 is a dominating vertex for $S_{a,b,c}$, a contradiction.

Therefore we assume that P_2 uses some edge $y_1y_2 \in E'_X$, $y_2 \in X'$. Note that there is only one edge of P_2 in E'_X as otherwise we get a shorter path because $c = y'$ is adjacent to all the vertices in X'. Now consider part of path P_2 from y_2 to z' and part of P_1 from x' to d and the path P_3 and edge dy_2. We get an induced cycle of length more than three or an AT S' in G. None of the vertices of this AT is in $D(a,b,c)$ and none of the edges of S' in E'_X. This means we get an AT in G such that it contains a vertex from $N[v_i]$, $7 \leq i \leq p-6$. Now we get a contradiction by Lemma 3.18 because at least one vertex of the AT, S' must be in $D(a,b,c)$ or S' is in $G[a,b,c]$, implying that $G[a,b,c]$ is not ripe.

We conclude that P_2 contains a vertex from $D(a,b,c)$.

Before we proceed we summarize the followings.

(1) $d_2 \in D(a,b,c)$. Since every vertex in $D(a,b,c)$ is adjacent to $c = y'$ and P_2 is the shortest path.

(2) x' is not adjacent to any vertex in $G[a,b,c]$. Otherwise by Lemma 3.10 $x'd_2$ is an edge.

(3) $P_1 \cap D(a,b,c) = \emptyset$.

(4) We may assume that $B[a,b] \cap P_1 \neq \emptyset$. Note that P_1 is a path from $x' \notin G[a,b,c]$ to vertex $d \in G[a,b,c]$. Therefore by Lemma 3 we may assume that $B[a,b] \cap P_1 \neq \emptyset$.

Let P'_2 be a path from d to d_2 and then following P_2 to z'.

Case 1. z' is adjacent to some v_j, $0 \leq j \leq p+1$.

We show that $j \leq i$. For contradiction suppose $j > i$. Now P_3 from z' to x' must contain a vertex from X'. Otherwise we would have a path v_jP_3Q outside the neighborhood of $y' = c$ where Q is part of P_1 from x' to v_1. But this would be a contradiction to X' is a separator.

We continue by having $j \leq i$. If $j \leq 3$ then we get an AT S with the vertices d, z', x' as follows: x' is joined with z' via part of P_3 from x' to the first time P_3 reaches to v_j and then to z' (note that since $d \in N[v_i]$, $7 \leq i \leq p-6$ then dv_j is not an edge). d is joined with z' via P'_2 and finally d is joined with x' via path P_1. We note that none of the edges S belongs to $E_{X'}$. However since $d_2 \in D(a,b,c)$ and $d \in N[v_i]$, $7 \leq i \leq p-6$ the conditions of the Lemma 3.18 are met while none of the (1,2,3,4) consequences of Lemma 3.18 holds and hence we get a contradiction to $S_{a,b,c}$ is ripe. If $j > 3$ then we get an AT, v_2, d, z' as follows: v_2 is joined wit d via part of P_1 from the neighborhood of v_2 to d. There is a path from x' to z' using the vertices $v_2, v_3, \ldots, v_j, z'$ and then z' to d via the vertices $v_j, v_{j+1}, \ldots, v_i, d$ yielding an AT, inside $G[a,b,c]$. This is a contradiction to $S_{a,b,c}$ is ripe.

Case 2. z' has no neighbor in the path $P_{a,b}$.
If the path $P_3 \cap G[a,b,c] = \emptyset$ then we obtain a smaller AT S with the vertices d, x', z' as follows. x' is joined with d via part of P_1 from x to d and P_3 joins x' to z' and P'_2 joins d_2 and z'. We note that none of the edges of S belongs to $E_{X'}$ and S contains a vertex d from $N[v_i]$, $7 \leq i \leq p-6$. Now the conditions of the Lemma 3.18 are met while none of the (1,2,3,4) consequences of Lemma 3.18 holds and hence we get a contradiction to $G[a,b,c]$ is an interval i.e. $S_{a,b,c}$ is ripe. Therefore

$P_3 \cap G[a,b,c] \neq \emptyset$.

Since $P_3 \cap D(a,b,c) = \emptyset$ and $P_3 \cap G[a,b,c] \neq \emptyset$, by Lemma 3 $P_3 \cap B[a,b] \cup E[a,b]$. Consider the first time that P_3 visits a vertex in $P_{a,b}$. Either we have $P_3 \cap E[a,b] \neq \emptyset$ or $P_3 \cap B[a,b] \neq \emptyset$. Recall that by our assumption $B[a,b] \cap P_1 \neq \emptyset$. If $B[a,b] \cap P_3 = \emptyset$ then $E[a,b] \cap P_3 \neq \emptyset$ then P_3 must contain a vertex from X'. Otherwise we would have a path $v_p P_3' P_1' v_1$ outside the neighborhood of $y' = c$ where P_3' is part of P_3 from a vertex in the neighborhood of v_1 to x' and P_1' is part of P_1 from x' to a vertex in the neighborhood of v_p. But this would be a contradiction to X' is a separator.

We continue by having $P_3 \cap B[a,b] \neq \emptyset$. Now consider the first time P_3 has a vertex from $N[v_1]$, and the first time P_1 contains a vertex from $N[v_1]$. We obtain a path from x' to z' that avoids the neighborhood of d. Now we get an AT d, x', z', and similarly we get a contradiction.

$\diamond$

Lemma 5.8 *Let G be a chordal graph without small ATs and let $S_{a,b,c}$ be a minimum ripe AT with the path $P_{a,b} = a, v_1, v_2, \ldots, v_p, b$. Let X be a minimum separator in $G \setminus N(c)$ that separates v_6 from v_{p-5} and it contains a vertex v_i, $7 \leq i \leq p-6$. Then there is a minimum set of fill-in edges F such that $G \cup F$ is an interval graph and at least one of the following holds:*

(i) If $S_{a,b,c}$ is of type (1) then F contains at least one fill-in edge from

$$\{bu, au, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

If $S_{a,b,c}$ is of type (2) then F contains at least one fill-in edge from

$$\{bu, aw, ca, cb, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

(ii) F contains all the edges av_i, $2 \leq i \leq p$ and ab.

(iii) F contains all the edges bv_i, $1 \leq i \leq p-1$ and ab.

(iv) F contains all the edges cf, $f \in G[a,b,c]$

(v) F contains all edges $E_X = \{c'x | c' \in C, x \in X\}$.

Proof: Let H be a minimum set of fill-in edges such that $G \cup H$ is an interval graph. If $S_{a,b,c}$ is of type (1) and H contains an edge from

$$\{bu, au, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

or when $S_{a,b,c}$ is of type (2) and H contains an edge from

$$\{bu, aw, ca, cb, cv_1, cv_2, cv_3, cv_4, cv_5, cv_6, cv_{p-5}, cv_{p-4}, cv_{p-3}, cv_{p-2}, cv_{p-1}, cv_p\}$$

then we set $F = H$.

Suppose H contains edge ab. We argue that H also contains all the edges av_i, $2 \leq i \leq p$ or all the edges bv_i, $1 \leq i \leq p-1$. If H contains edge ab then we may assume that there exists a minimum AT, $S_{x,y,z}$ such that ab is a fill-in edge for $S_{x,y,z}$. By Lemma 5.5 (1,2) ab is a cross

fill-in edge for $S_{x,y,z}$ and up to symmetry we have $a = x$ and $b = u'$ and for every $x' \in G[a, b, c]$, $S_{x',y,z}$ is an AT with the same number of vertices as $S_{x,y,z}$. This implies that every optimal solution must also add the edges $u'v_1, u'v_2, \dots, u'v_{p-1}$ and in particular H contains all the edges bv_i, $1 \leq i \leq p - 1$ and ab. In this case we also set $F = H$. If one if the items (3) and (4) of the Lemma 5.5 holds again we conclude that H must contains all the edges bv_i, $1 \leq i \leq p - 1$ and ab or H must contain all the edges av_i, $2 \leq i \leq p$ and ab.

We will proceed by assuming that H does not contain any of the edges au, bu (aw, bu if of type 2), ab, cv_r, $0 \leq r \leq 6$, and cv_r, $p - 5 \leq r \leq p + 1$. Moreover we assume that H does not contain all the edges cf, $f \in G[a, b, c]$ as otherwise we set $F = H$.

Let $W = \{w|cw \in H\}$ be the set of vertices adjacent to c via fill-in edges. Because $S_{a,b,c}$ is an AT there is no path from v_6 to v_{p-5} in $(G \cup H) \setminus N(c)$. Hence, set W should contain a *minimal* v_6, v_{p-5}-separator X' in $G \setminus N(c)$, containing a vertex v_i, $7 \leq i \leq p - 6$.

Claim 5.9 *Let c' is a vertex adjacent to c and not adjacent to any vertex on the path $P_{a,b}$ in G. Then H contains all the edges $c'x'$, $x' \in X'$.*

Proof: Indeed, by Lemma 3.4, c' is adjacent to u, and thus $S_{a,b,c'}$ is also a minimum AT with the same number of vertices as $S_{a,b,c}$. Since $S_{a,b,c'}$ is no longer an AT in $G \cup H$ and none of the au, bu is an edge in H and ab is not an edge in H we have that H contains at least one edge $c'v_j$. Let us assume that $v_j \neq v_i$ is the closest vertex to v_i such that cv_j is not in H. (Observe that we assumed that the H is a minimal set of fill edges such that $G \cup H$ is an interval graph and cv_i is in H since $S_{a,b,c}$ is an AT in G). Let P be part of $P_{a,b}$ from v_i to v_j. By our assumption for H, P has no chords. No vertex of P except v_j and v_i is adjacent to c or c'. Thus the cycle c, P, c', c' is a chordless cycle in $G \cup H$, which is a contradiction. Therefore we conclude that $c'v_i$ is an edge. No let $W' = \{w'|c'w' \in H\}$. Because $S_{a,b,c'}$ is an AT and none of the au, bu, cv_r, $v_r \in \{v_1, \dots, v_6, v_{p-5}, \dots, v_p\}$ is in H (note that $7 \leq i \leq p - 6$) and there is no path from v_6 to v_{p-5} in $(G \cup H) \setminus N(c)$. Hence, set W' should contain a *minimal* v_6, v_{p-5}-separator X'' containing $v_i = v_j$ in $G \setminus N(c)$. Because $G[a, b, c]$ is an interval graphs and both X' and X'' contain v_i we have $X' = X''$. $\diamond$

Now by applying the Claim 5.9 for every $c'' \in C$ we conclude that $c''x'$, $x' \in X'$ is in H. Let $E'_X = \{c'x'|c' \in C, x' \in X'\}$. We observe that X' is a clique because $G[a, b, c]$ is an interval graph.

We define $F = (H \setminus E'_X) \cup E_X$. Let us note that because none of the sets E_X and E'_X contains edges of G and because X is a minimum separator, we have that $|F| \leq |H|$. In what follows, we prove that $I = G \cup F$ is an interval graph. For a sake of contradiction, let us assume that I is not an interval graph. We note that by Lemma 5.7 adding the edges $c'x'$, $c' \in C$ and $x' \in X'$ would not add new AT in G. Therefore by Theorem 1.1 we may assume that I contains an induced cycle of length more than three or a big AT.

Case 1. I contains large AT.

Let $S_{x,y,z}$ be an AT in I. We assume that vertices x and y are connected in $S_{x,y,z}$ by an induced path $x, w_1, w_2, \dots, w_q, y$, where $q \geq 6$. Because $S_{x,y,z}$ is not an AT in $G \cup H$, set $E'_X \setminus E_X$ must contain some fill-in edge cd, for $S_{x,y,z}$. By definition of $G[a, b, c]$, d is adjacent to some v_i, $3 \leq i \leq p - 2$. Every fill-in edge of $S_{x,y,z}$ is either long, cross, or bottom, see Definition 5.3.

Claim A. *cd is not a cross fill-in edges of $S_{x,y,z}$.*
By Lemma 5.4 the fill-in edge cd is a long fill-in edge of $S_{x,y,z}$ and not a cross fill-in edge.

Claim B. *cd is not a bottom fill-in edges of $S_{x,y,z}$.*
For contradiction suppose cd is a bottom fill-in edge for $S_{x,y,z}$. Thus we have $c = x$ and $y = d$ or $c = y$ and $x = d$. W.l.o.g assume that $c = x$ and $y = d$. Now there is a path $Q = c, w_1, w_2, \ldots, w_q, d$ from c to d. Since Q is chordless, by Lemma 3.12 w_1 is a dominating vertex for $S_{a,b,c}$ and by Corollary 3.9 $w_1 d$ is an edge. This is a contradiction because $q > 2$.

We conclude that cd is a long fill-in edges for $S_{x,y,z}$. Now we have $z = c$ or $z = d$. If $z = d$ then by Lemma 5.4 (2) for every vertex $f \in G[a, b, c]$, $S_{x,y,f}$ is an AT with the same path $x, w_1, w_2, \ldots, w_q, y$. Since H does not contain any of the edge au, bu, cv_i, $i \in \{1, \ldots, 6, p-5, \ldots, p\}$, and $S_{a,b,f}$ is an AT, H must contain the edge cf. But this is a contradiction as we assumed that H does not contain all the edges cf, $f \in G[a, b, c]$.

Therefore we suppose $z = c$. We argue that there exists a fill-in edge $cd' \in E_X$, such that $d' \in \{x, w_1, w_2, \ldots, w_q, y\}$. Since $z = c$, Lemma 5.4 (1) implies that $P_{x,y} \cap B[a, b] \neq \emptyset$ and $P_{x,y} \cap E[a, b] \neq \emptyset$ and every v_i, $2 \leq i \leq p - 1$ has a neighbor in $P_{x,y}$. Therefore there would be a path from v_2 to v_{p-1}. This is a contradiction to X being a v_6, v_{p-5} separator and hence there exists some fill-in edge $cd' \in E_X$, such that $d' \in \{x, w_1, w_2, \ldots, w_q, y\}$.

Case 2. I contains a cordless cycle of length more than three. This implies that there exists a path Q from c to $d \in G[a, b, c]$. However by Lemma 3.12 the second vertex of Q say d' is a dominating vertex for $S_{a,b,c}$ and since $d \in G[a, b, c]$ by Corollary 3.9 $d'd$ is an edge. This implies that the length of Q is 2, a contradiction.

$\diamond$

In the following Lemma we address the correctness and complexity of the Algorithm.

Theorem 5.10 *The Branching Algorithm is optimal and its running time is $O(c^k n(n+m))$, $c \in \min\{17, k\}$ for parameter k.*

Proof: The correctness of the Branching Algorithm is justified by Lemma 5.8. By Lemma 5.8(1,2) there are five ways of adding one fill in edge for AT $S_{a,b,c}$ of type 1 (type 2). Either we add one of the edges au, bu, cv_i, $1 \leq i \leq 6$ or $p - 5 \leq i \leq p$ or we add at least one edge from c to set X or we add edge ab and hence the Algorithm needs to add $p - 2$ other fill in edges. Note that once we add ab then the parameter k decrease by at least 5. In order to get the maximum number of branching we may assume that no bottom fill-in edge ab is added. By looking at the small AT, together with the AT's of type 1 and type 2 there are at most $\max\{17, k\}$ possible ways to add a fill in edge to $S_{a,b,c}$ and at each step the parameter k is decreased by at least one. We may deploy the algorithms developed in [14] with the running time $O(n(n+m))$ to find ATs. Therefore the running time of the algorithm is $O(c^k n(n+m))$, $c \in \min\{17, k\}$. $\diamond$

6 Conclusion and future work

We have shown that there exist single exponential FPT algorithms for k-interval deletion problem. The obstruction for the class of interval graphs is not finite but the obstructions can be partitioned

into a constant number of families.

Let Π be a class of graphs. We say Π has *family bounded* property if the forbidden subgraphs for this class can be partitioned into a constant number of families. Let $\Pi + kv$ denotes the problem of deleting k vertices (edges) from (into) input graph G such that the resulting graphs becomes a member of Π. It would be interesting to study the following problem.

Problem 6.1 *For which classes* Π *of graphs with family bounded property, the problem* $\Pi + kv$ *is FPT ?*

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